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LOGISTIC DIRICHLET PROBLEMS WITH DISCONTINUOUS COEFFICIENTS

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Dedicated to Professor Jean-Michel Bony on the occasion of his 60th birthday

ABSTRACT. This paper is devoted to the study of the existence of positive solutions of semilinear Dirichlet eigenvalue problems for diffusive logistic equations with *discontinuous* coefficients which model population dynamics in environments with spatial heterogeneity. The approach here is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of *singular integral operators*. Moreover, we make use of an L^p variant of an estimate for the Green operator of the Dirichlet problem introduced in the study of Feller semigroups.

1. Introduction and Main Results

Let Ω be a bounded domain of \mathbf{R}^N , $N \geq 3$, with boundary $\partial\Omega$ of class $C^{1,1}$. In this paper we consider a second-order, uniformly elliptic differential operator with *discontinuous* coefficients of the form

$$\mathcal{L}u := - \sum_{i,j=1}^N a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b^i(x) \frac{\partial u}{\partial x_i} + c(x)u.$$

Here:

(1) $a^{ij}(x) \in \text{VMO} \cap L^\infty(\mathbf{R}^N)$, $a^{ij}(x) = a^{ji}(x)$ for almost all $x \in \Omega$, and there exists a constant $a_0 > 0$ such that

$$a_0^{-1}|\xi|^2 \leq \sum_{i,j=1}^N a^{ij}(x)\xi_i\xi_j \leq a_0|\xi|^2 \quad \text{for almost all } x \in \Omega \text{ and all } \xi \in \mathbf{R}^N.$$

(2) $b^i(x) \in L^\infty(\Omega)$.

(3) $c(x) \in L^\infty(\Omega)$ and $c(x) \geq 0$ for almost all $x \in \Omega$.

This paper is devoted to the study of the existence of positive solutions of the following *logistic Dirichlet problem* with indefinite weight:

$$\begin{cases} \mathcal{L}u = \lambda(m(x)u - h(x)u^2) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

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Here:

- (1) λ is a positive parameter.
- (2) $m(x) \in C(\overline{\Omega})$ and $m(x)$ may *change sign* in Ω .
- (3) $h(x) \in C(\overline{\Omega})$ and $h(x) > 0$ on $\overline{\Omega}$.

We discuss our motivation and some of the modeling process leading to problem (1.1). The basic interpretation of the various terms in problem (1.1) is that the solution $u(x)$ represents the population density of a species inhabiting a region Ω . The members of the population are assumed to move about Ω via the type of random walks occurring in Markovian motion which is modeled by the diffusive term $(1/\lambda)\mathcal{L}$; hence $1/\lambda$ represents the rate of diffusive dispersal, so large values of $1/\lambda$ the population spreads more rapidly than for small values of $1/\lambda$. The local rate of change in the population density is described by the density dependent term $m(x) - h(x)u$. In this term, the function $m(x)$ describes the rate at which the population would grow or decline at the location x in the absence of crowding or limitations on the availability of resources. The sign of $m(x)$ will be positive on favorable habitats for population growth and negative on unfavorable ones. Specifically, the function $m(x)$ may be considered as a food source or any resource that will be good in some areas and bad in others. The term $-h(x)u$ describes the effects of crowding on the growth rate of the population at the location x ; these effects are assumed to be independent of those determining the growth rate. The size of $h(x)$ describes the strength of the effects of crowding within the population.

On the other hand, in terms of biology, the homogeneous Dirichlet condition represents that Ω is surrounded by a completely hostile exterior such that any member of the population which reaches the boundary dies immediately; in other words, the exterior of the domain is deadly to the population.

The purpose of this paper is to generalize two main results, Theorems 1 and 2, of Hess and Kato [12] to the VMO case. More precisely, we discuss the changes that occur in the global structure of positive solutions as a parameter λ varies from the *principal eigenvalue* $\lambda_1(m)$ of the linearized Dirichlet eigenvalue problem

$$\begin{cases} \mathcal{L}u = \lambda m(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

The next theorem plays an essential role in the study of Dirichlet problem (1.2) with an indefinite weight function (see Theorem 1.2 below):

Theorem 1.1. *Let $N < p < \infty$. We define a linear operator*

$$L : C_0(\overline{\Omega}) \longrightarrow C(\overline{\Omega})$$

as follows:

- (a) *The domain $D(L)$ of definition is the set*

$$D(L) = \left\{ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \mathcal{L}u \in C(\overline{\Omega}) \right\}.$$

- (b) *$Lu = \mathcal{L}u$ for all $u \in D(L)$.*

Here

$$C_0(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\}.$$

Then we have the following two assertions:

- (i) The operator L is densely defined and closed.
- (ii) The operator $L : D(L) \rightarrow C(\overline{\Omega})$ is an algebraic and topological isomorphism, where the domain $D(L)$ is equipped with the graph norm.

Remark 1.1. It is easy to verify that the domain $D(L)$ is independent of p , for all $N < p < \infty$ (see [26, the proof of Lemma 4.2]).

To study the logistic Dirichlet eigenvalue problem (1.1), we introduce two ordered Banach spaces and their positive cones associated with the operator L in the following way: We let

$$Y = C(\overline{\Omega}),$$

$$P_Y = \{v \in C(\overline{\Omega}) : v \geq 0 \text{ in } \Omega\},$$

and

$$X = D(L) = \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \mathcal{L}u \in C(\overline{\Omega})\},$$

$$P_X = \{u \in D(L) : u \geq 0 \text{ in } \Omega\}$$

$$= \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \mathcal{L}u \in C(\overline{\Omega}), u \geq 0 \text{ in } \Omega\}.$$

Here it should be noticed that we have, by Sobolev's imbedding theorem,

$$X = D(L) \subset C_0^1(\overline{\Omega}),$$

since $N < p < \infty$ and so $2 - N/p > 1$.

Let $M : Y \rightarrow Y$ be the multiplication operator by a function $m(x) \in C(\overline{\Omega})$. A function $u \in X \setminus \{0\}$ is called an eigenfunction of problem (1.2) if it satisfies the equation

$$Lu = \lambda Mu \quad \text{in } Y. \tag{1.3}$$

We are interested in the existence of non-zero eigenvalues having a positive eigenfunction. It should be emphasized that if the weight function $m(x)$ does not change sign in Ω , say, if $m(x) > 0$ on $\overline{\Omega}$, then the Kreĭn and Rutman theorem asserts that the *spectral radius* $\text{spr}(L^{-1}M) := \lim_{n \rightarrow \infty} \|(L^{-1}M)^n\|^{1/n}$ is the only eigenvalue of $L^{-1}M$ whose associated eigenspace contains a positive eigenfunction. In this paper we consider the case where the weight function $m(x) \in C(\overline{\Omega})$ may change sign in Ω .

Our first main result is a generalization of Hess and Kato [12, Theorem 1] and Hess [11, Theorem 16.1] to the *VMO* case:

Theorem 1.2. *If $m(x) \in C(\overline{\Omega})$ is positive somewhere in Ω , then the Dirichlet eigenvalue problem (1.2) admits a unique positive eigenvalue $\lambda_1(m)$ with a positive eigenfunction $\varphi_1 \in \text{Int}(P_X)$. Moreover, the eigenvalue $\lambda_1(m)$ has the following two properties:*

- (i) *If $\hat{\lambda} \in \mathbf{C}$ is an eigenvalue of the equation $Lv = \hat{\lambda}Mv$ obtained by the complexification of the equation $Lv = \lambda Mv$ and if $\text{Re } \hat{\lambda} > 0$, then we have $\text{Re } \hat{\lambda} \geq \lambda_1(m)$.*
- (ii) *The reciprocal $\mu_1(m) := 1/\lambda_1(m)$ is an eigenvalue of the operator $L^{-1}M : X \rightarrow X$ with algebraic multiplicity one.*

A pair $(\lambda, u) \in \mathbf{R} \times X$ is called a *positive solution* of the logistic Dirichlet eigenvalue problem (1.1) if $\lambda > 0$ and $u \in P_X \setminus \{0\}$ and if the pair (λ, u) satisfies the operator equation

$$Lu = \lambda F(u) \quad \text{in } Y, \tag{1.4}$$

where $F(u)$ is the *Nemytskii operator* associated with the term $m(x)u - h(x)u^2$:

$$F(u)(x) = m(x)u(x) - h(x)u(x)^2, \quad x \in \overline{\Omega}.$$

Our second main result is a generalization of Hess and Kato [12, Theorem 2] and Hess [11, Theorem 27.1] to the *VMO* case:

Theorem 1.3. *Assume that $m(x) \in C(\overline{\Omega})$ is positive somewhere in Ω . Then we have the following four assertions:*

(i) *If a pair $(\lambda, u) \in \mathbf{R} \times X$ is a positive solution of problem (1.1), then it follows that $\lambda > \lambda_1(m)$.*

(ii) *There is an unbounded arc \mathcal{C}^+ of positive solutions (λ, u) of problem (1.1) emanating from $(\lambda_1(m), 0)$, and the point $(\lambda_1(m), 0)$ is the only bifurcation point for positive solutions from the line of trivial solutions.*

(iii) *There is a continuous map $\tilde{u}(\cdot) : [\lambda_1(m), \infty) \rightarrow P_X$, with $\tilde{u}(\lambda_1(m)) = 0$, such that $\mathcal{C}^+ = \{(\lambda, \tilde{u}(\lambda)) : \lambda_1(m) \leq \lambda < \infty\}$. Moreover, the map $\tilde{u}(\cdot)$ is continuously differentiable in the interval $(\lambda_1(m), \infty)$.*

(iv) *The $\tilde{u}(\lambda)$ are uniformly bounded for all $\lambda > \lambda_1(m)$:*

$$\max_{\overline{\Omega}} |\tilde{u}(\lambda)| \leq \frac{\max_{\overline{\Omega}} m}{\min_{\overline{\Omega}} h}. \quad (1.5)$$

Our situation may be represented schematically by the following bifurcation diagram:

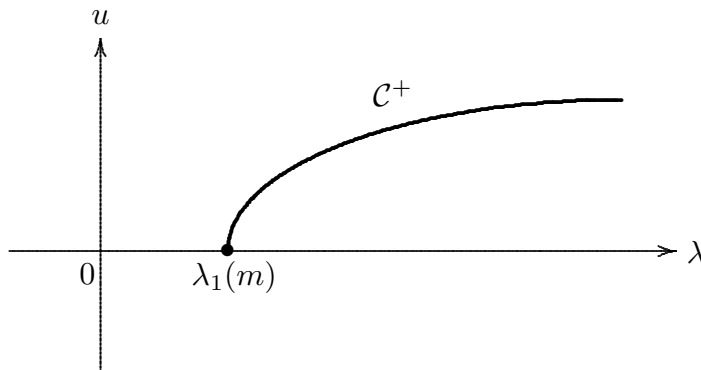


Figure 1.1

Rephrased, Theorem 1.3 asserts that the models we consider predict persistence for a population if its diffusion rate $1/\lambda$ is below the critical value $1/\lambda_1(m)$ depending on the coefficient $m(x)$ which describes the growth rate.

The rest of this paper is organized as follows. In Section 2 we summarize some important topics from real analysis and functional analysis. These topics form a necessary background for the proof of Theorems 1.1, 1.2 and 1.3. Section 3 is devoted to the proof of Theorem 1.1. We make essential use of an existence and uniqueness theorem for the Dirichlet problem with VMO coefficients (Theorem 3.2), which is proved in Chiarenza, Frasca and Longo [4] and [5] by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of singular integral operators. Moreover, the proof of the density of the domain $D(L)$

is based on an L^p variant of an estimate for the Green operator of the Dirichlet problem proved in Taira [25] in the study of Feller semigroups. In Sections 4 and 5 we prove Part (i) and Part (ii) of Theorem 1.2, respectively, by using Theorem 1.1 and the Kreĭn and Rutman theorem (Theorem 2.4). The proof of Theorem 1.2 can be accomplished in a series of lemmas, just as in Hess and Kato [12]. Section 6 is devoted to the proof of Theorem 1.3. By Theorems 1.1 and 1.2, we can apply the Crandall and Rabinowitz bifurcation theory (Theorem 2.5) to the logistic Dirichlet problem, just as in Hess and Kato [12] and also Hess [11]. In the final Section 7 we give some remarks concerning the logistic Neumann problem.

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2. Preliminaries

This section is devoted to a review of some important topics from real analysis and functional analysis that form a necessary background for the proof of Theorems 1.1, 1.2 and 1.3.

2.1 BMO and VMO functions.

In this subsection we recall some basic definitions and results concerning BMO and VMO functions in \mathbf{R}^N . For more thorough treatments of this subject, the reader might be referred to Garnett [9] and Torchinsky [27].

A function $f(x) \in L^1_{\text{loc}}(\mathbf{R}^N)$ is said to be of *bounded mean oscillation*, $f(x) \in \text{BMO}$, if it satisfies the condition (see John and Nirenberg [13])

$$\|f\|_* := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty,$$

where the supremum is taken over all balls B in \mathbf{R}^N and f_B is the average of f over B

$$f_B := \frac{1}{|B|} \int_B f(x) dx.$$

It should be noticed that the quantity $\|f\|_*$ defines a norm on the quotient space BMO/\mathbf{R} .

Next we introduce a subspace of BMO functions whose BMO norm over a ball vanishes as the radius of the ball tends to zero. More precisely, if $f(x) \in \text{BMO}$ and $r > 0$, then we let

$$\eta(r) := \sup_{\rho \leq r} \frac{1}{|B|} \int_B |f(x) - f_B| dx,$$

where the supremum is taken over all balls B with radius $\rho \leq r$.

A function $f(x) \in \text{BMO}$ has *vanishing mean oscillation*, $f(x) \in \text{VMO}$, if it satisfies the condition (see Sarason [21])

$$\lim_{r \downarrow 0} \eta(r) = 0.$$

The function $\eta(r)$ is called the VMO modulus of f .

The assumption $a^{ij}(x) \in \text{VMO}$ means a kind of continuity in the average sense, not in the pointwise sense. This property implies that VMO functions may be approximated by smooth functions. The next proposition collects some important results concerning VMO functions:

- Proposition 2.1.** (i) If $f(x) \in \text{VMO}$, then, for any $\varepsilon > 0$ there exists a uniformly continuous function $g_\varepsilon(x)$ on \mathbf{R}^N such that $\|f - g_\varepsilon\|_* < \varepsilon$.
(ii) Uniformly continuous functions that belong to BMO are VMO functions.
(iii) $W^{\theta, N/\theta}(\mathbf{R}^N) \subset \text{VMO}$, for $0 < \theta \leq 1$.

Examples 2.2. (i) $\ln|x| \in \text{BMO}$, but $\ln|x| \notin \text{VMO}$.

(ii) $\ln|\ln|x|| \in \text{VMO}$.

(iii) If Ω is the ball of \mathbf{R}^N with radius one about the origin, then the function $\sin(\ln(\ln(\frac{4}{|x|})))$ is in $W^{1,N}(\Omega) \cap L^\infty(\Omega)$ and so in $\text{VMO}(\Omega)$, but not in $C(\overline{\Omega})$.

2.2 The Kreĭn and Rutman theorem.

In this subsection we recall some basic definitions and results concerning ordered Banach spaces. For more thorough treatments of this subject, the reader might be referred to Amann [1].

Let X be a non-empty set. An ordering \leq in X is a relation in X that is reflexive, transitive and antisymmetric. A non-empty set together with an ordering is called an ordered set.

Let V be a real vector space. An ordering \leq in V is said to be *linear* if the following two conditions are satisfied:

- (i) If $x, y \in V$ and $x \leq y$, then we have $x + z \leq y + z$ for all $z \in V$.
- (ii) If $x, y \in V$ and $x \leq y$, then we have $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

A real vector space together with a linear ordering is called an *ordered vector space*.

If $x, y \in V$ and $x \leq y$, then the set $[x, y] = \{z \in V : x \leq z \leq y\}$ is called an *order interval*.

If we let

$$Q = \{x \in V : x \geq 0\},$$

then it is easy to verify that the set Q satisfies the following two conditions:

- (iii) If $x, y \in Q$, then $\alpha x + \beta y \in Q$ for all $\alpha, \beta \geq 0$.
- (iv) If $x \neq 0$, then at least one of x and $-x$ does not belong to Q , or equivalently, $Q \cap (-Q) = \{0\}$.

The set Q is called the *positive cone* of the ordering \leq .

Let E be a Banach space E with a linear ordering \leq . The Banach space E is called an *ordered Banach space* if the positive cone P is closed in E . It is to be expected that the topology and the ordering of an ordered Banach space are closely related if the norm is *monotone*: If $0 \leq x \leq y$, then $\|x\| \leq \|y\|$.

For $x, y \in E$, we write

$$\begin{aligned} x &\geq y && \text{if } x - y \in P, \\ x &> y && \text{if } x - y \in P \setminus \{0\}. \end{aligned}$$

If the interior $\text{Int}(P)$ is non-empty, then we write

$$x \gg y \quad \text{if } x - y \in \text{Int}(P).$$

Example 2.3. Let $Y := C(\overline{\Omega})$. For two functions $u, v \in Y$, we write $u \leq v$ if $u(x) \leq v(x)$ for all $x \in \overline{\Omega}$. Then it is easy to verify that the space Y is an ordered Banach space with the linear ordering \leq and the positive cone

$$P_Y = \{u \in C(\overline{\Omega}) : u \geq 0 \text{ on } \overline{\Omega}\},$$

with non-empty interior

$$\text{Int}(P_Y) = \{u \in C(\overline{\Omega}) : u > 0 \text{ on } \overline{\Omega}\}.$$

A linear operator $A : E \rightarrow E$ is said to be *strongly positive* if Ax is an interior point of P for every $x \in P \setminus \{0\}$:

$$x > 0 \implies Ax \gg 0.$$

Then the Kreĭn and Rutman theorem reads as follows (see Kreĭn and Rutman [16]):

Theorem 2.4. *Let (E, P) be an ordered Banach space with non-empty interior $\text{Int}(P)$. Assume that $K : E \rightarrow E$ is strongly positive and compact. Then we have the following three assertions:*

(i) *The spectral radius $r = \text{spr}(K) := \lim_{n \rightarrow \infty} \|K^n\|^{1/n}$ is positive, and r is a unique eigenvalue of K having positive eigenfunction $x \in \text{Int}(P)$. The eigenvalue r is algebraically simple.*

(ii) *Moreover, r is also an algebraically simple eigenvalue of the adjoint operator $K^* : E^* \rightarrow E^*$, with positive eigenfunction $x^* \in \text{Int}(P^*)$. Here*

$$P^* = \{x^* \in E^* : \langle x^*, x \rangle \geq 0 \text{ for all } x \in P\}.$$

(iii) *Finally, we have $|\lambda| < r$ for all $\lambda \in \sigma(K)$ with $\lambda \neq r$, where $\sigma(K)$ is the spectrum of K .*

The eigenvalue r is called the *principal eigenvalue* of K .

2.3 Local bifurcation theory.

This subsection is devoted to local static bifurcation theory from a simple eigenvalue essentially due to Crandall and Rabinowitz [7]. For detailed studies of bifurcation theory, the reader is referred to Chow and Hale [6] and Nirenberg [19].

2.3.1 Differentiability. Let X, Y be Banach spaces, U an open set in X and $f : U \rightarrow Y$ a map. We say that the map f is (Fréchet) *differentiable at a point* $x \in U$ if there exist a continuous linear operator $A : X \rightarrow Y$ and a map ψ defined for h sufficiently small in X , with values in Y , such that

$$\begin{cases} f(x+h) = f(x) + Ah + \|h\|\psi(h), \\ \lim_{h \rightarrow 0} \psi(h) = 0. \end{cases}$$

It should be noticed that the continuous linear operator A is uniquely determined by f and x . The operator A is called the (Fréchet) *derivative* of f at x , and is denoted by $Df(x)$ or $f'(x)$. A map f is said to be (Fréchet) *differentiable on* U if it is (Fréchet) differentiable at every point of U . In this case the derivative Df

is a map of U into the Banach space $B(X, Y)$ of all continuous (bounded) linear operators:

$$\begin{aligned} Df : U &\longrightarrow B(X, Y) \\ u &\longmapsto Df(u). \end{aligned}$$

If, in addition, Df is continuous from U into $B(X, Y)$, we say that f is of class C^1 .

If the derivative Df is differentiable at a point $x \in U$ (resp. in U), we say that f is *twice differentiable* at x (resp. in U). The derivative of Df at x is called the *second derivative* of f at x , and is denoted by $D^2f(x)$. This is an element of the Banach space $B(X, B(X, Y))$ which can be naturally identified with the space $B_2(X, Y) = B(X, X; Y)$ of all continuous bilinear mappings of $X \times X$ into Y . A map $f : U \rightarrow Y$ is said to be of class C^2 in U if the derivatives Df and D^2f exist and are continuous in U .

Now we assume that the Banach space X is the product space of two Banach spaces X_1 and X_2 :

$$X = X_1 \times X_2.$$

For each point $x = (x_1, x_2) \in U \subset X$, we can consider the partial mappings

$$\begin{aligned} F_1 : u_1 &\longmapsto f(u_1, x_2), \\ F_2 : u_2 &\longmapsto f(x_1, u_2) \end{aligned}$$

of open subsets of X_1 and X_2 respectively into Y . We say that f is *differentiable with respect to the first* (resp. *second*) *variable* if the mapping $F_1(u_1)$ (resp. $F_2(u_2)$) is differentiable at x_1 (resp. at x_2). The derivative $DF_1(x_1)$ (resp. $DF_2(x_2)$) is an element of the Banach space $B(X_1, Y)$ (resp. $B(X_2, Y)$), and is called the *partial (Fréchet) derivative* of f at (x_1, x_2) with respect to the first (resp. second) variable. We write

$$\begin{aligned} f_{x_1}(x_1, x_2) &= DF_1(x_1), \\ f_{x_2}(x_1, x_2) &= DF_2(x_2). \end{aligned}$$

2.3.2 Bifurcation from a simple eigenvalue. Let $F(t, x)$ be a mapping of a neighborhood of $(0, 0)$ in a Banach space $\mathbf{R} \times X$ into a Banach space Y . Assume that there is a curve Γ in the space $\mathbf{R} \times X$ given by $\Gamma = \{w(t) : t \in I\}$, where I is an interval, such that $F(w) = 0$ for all $w \in \Gamma$. If there is a number $\tau_0 \in I$ such that every neighborhood of $w(\tau_0)$ contains zeros of F not lying on Γ , then the point $w(\tau_0)$ is called a *bifurcation point* for the equation $F(w) = 0$ with respect to the curve Γ . In many situations the curve Γ is of the form $\{(t, 0) : t \in \mathbf{R}, 0 \in X\}$. The basic problem of bifurcation theory is that of finding the bifurcation points for the equation $F(t, x) = 0$ with respect to Γ and studying the structure of $F^{-1}\{0\}$ near such points.

The next theorem, due to Crandall and Rabinowitz [7], gives sufficient conditions in order that the point $(0, 0)$ be a bifurcation point for the equation $F(t, x) = 0$ (see [7, Theorem 1.7]; [19, Theorem 3.2.2]; [6, Chapter 6, Theorem 6.1]):

Theorem 2.5. *Let X, Y be Banach spaces, and let V be a neighborhood of 0 in X and let $F : (-1, 1) \times V \rightarrow Y$ have the following four properties:*

- (1) $F(t, 0) = 0$ for $|t| < 1$.
- (2) The partial Fréchet derivatives F_t , F_x and F_{tx} of F exist and are continuous.
- (3) $\dim N(F_x(0, 0)) = \text{codim } R(F_x(0, 0)) = 1$.
- (4) If $N(F_x(0, 0)) = \text{span}[x_0]$, then $F_{tx}(0, 0)x_0 \notin R(F_x(0, 0))$.

If Z is a complement of $N(F_x(0, 0))$ in X , that is, if it is a closed subspace of X such that

$$X = N(F_x(0, 0)) \oplus Z,$$

then there exist a neighborhood U of $(0, 0)$ in $\mathbf{R} \times X$ and an open interval $(-a, a)$ such that the set of solutions of $F(t, x) = 0$ in U consists precisely of two continuous curves Γ_1 and Γ_2 which may be parametrized by t and α as follows (see Figure 2.1):

$$\begin{aligned} \Gamma_1 &= \{(t, 0) : (t, 0) \in U\}, \\ \Gamma_2 &= \{(\varphi(\alpha), \alpha x_0 + \alpha \psi(\alpha)) : |\alpha| < a\}. \end{aligned}$$

Here

$$\begin{aligned} \varphi : (-a, a) &\rightarrow \mathbf{R}, & \varphi(0) &= 0, \\ \psi : (-a, a) &\rightarrow Z, & \psi(0) &= 0. \end{aligned}$$

If, in addition, the partial Fréchet derivative F_{xx} is also continuous, then the functions φ and ψ are once continuously differentiable.

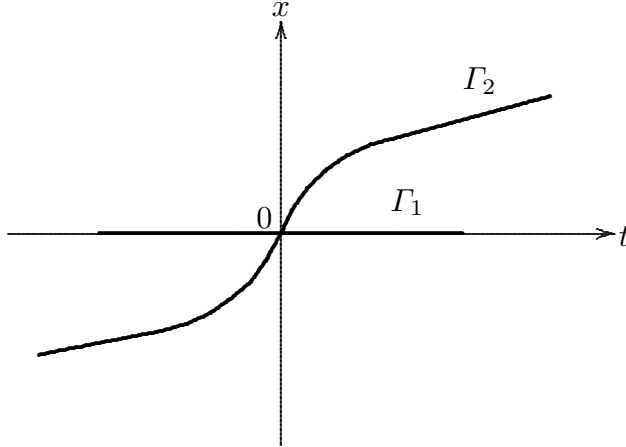


Figure 2.1

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Our proof is based on an existence and uniqueness theorem for the Dirichlet problem with VMO coefficients. To prove the *density* of the domain $D(L)$, we make use of an L^p variant of an estimate for the Green operator of the Dirichlet problem introduced in Taira [25] in the study of Feller semigroups.

3.1 The Dirichlet problem.

In this subsection we consider the Dirichlet problem in the framework of Sobolev spaces of L^p style.

If $1 < p < \infty$, we define the Sobolev space

$W^{2,p}(\Omega)$ = the space of (equivalence classes of) functions
 $u \in L^p(\Omega)$ whose derivatives $D^\alpha u$, $|\alpha| \leq 2$, in the
 sense of distributions are in $L^p(\Omega)$,

and the space

$B^{2-1/p,p}(\partial\Omega)$ = the space of the boundary values $\gamma_0 u$ of functions
 $u \in W^{2,p}(\Omega)$.

In the space $B^{2-1/p,p}(\partial\Omega)$, we introduce a norm

$$|\varphi|_{B^{2-1/p,p}(\partial\Omega)} = \inf \{ \|u\|_{W^{2,p}(\Omega)} : u \in W^{2,p}(\Omega), \gamma_0 u = \varphi \}.$$

More precisely, it is known (cf. [2], [28]) that the space $B^{2-1/p,p}(\partial\Omega)$ is a Besov space.

Our starting point is the following existence and uniqueness theorem for the Dirichlet problem with *VMO* coefficients (cf. [3, Théorème 3]):

Theorem 3.1. *Let $N < p < \infty$ and $\alpha \geq 0$. Then the non-homogeneous Dirichlet problem*

$$\begin{cases} (\mathcal{L} + \alpha)u = f & \text{in } \Omega, \\ \gamma_0 u = \varphi & \text{on } \partial\Omega \end{cases} \quad (3.1)$$

has a unique solution $u \in W^{2,p}(\Omega)$ for any $f \in L^p(\Omega)$ and any $\varphi \in B^{2-1/p,p}(\partial\Omega)$. In particular, we have the a priori estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C \left(\|(\mathcal{L} + \alpha)u\|_{L^p(\Omega)} + |\gamma_0 u|_{B^{2-1/p,p}(\partial\Omega)} \right), \quad (3.2)$$

with a constant $C = C(\alpha) > 0$, independent of u .

If we associate with problem (3.1) a continuous linear operator

$$\mathcal{A}(\alpha) = (\mathcal{L} + \alpha, \gamma_0) : W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B^{2-1/p,p}(\partial\Omega),$$

then Theorem 3.1 asserts that the mapping $\mathcal{A}(\alpha)$ is an algebraic and topological *isomorphism*. Indeed, the continuity of the inverse of $\mathcal{A}(\alpha)$ follows immediately from an application of Banach's closed graph theorem.

3.2 Proof of Theorem 3.1.

Step 1: Our proof is based on the following existence and uniqueness theorem for the *homogeneous* Dirichlet problem due to Chiarenza, Frasca and Longo [5, Theorems 4.3 and 4.4]:

Theorem 3.2. *Let*

$$\mathcal{L}_0 u = - \sum_{i,j=1}^N a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Then, for any $f \in L^p(\Omega)$ with $1 < p < \infty$ there exists a unique solution $u \in W^{2,p}(\Omega)$ of the Dirichlet problem

$$\begin{cases} \mathcal{L}_0 u = f & \text{in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.3)$$

The proof of Theorem 3.2 is given in detail in Chiarenza, Frasca and Longo [4] and [5]. The proof is based on some interior and boundary estimates for the solutions of problem (3.3) which requires the VMO assumption on the coefficients. From these estimates, an *a priori* estimate follows. Since VMO functions can be approximated by smooth functions (Proposition 2.1), we can prove the existence result of problem (3.3) in a standard way if we approximate the operator \mathcal{L}_0 with similar operators with smooth coefficients. Both the interior and boundary estimates are consequences of explicit representation formulas for the solutions of problem (3.3) and also of the L^p -boundedness of Calderón–Zygmund *singular integral operators* appearing in those representation formulas. For the uniqueness result of problem (3.3), we make essential use of the Bakel'man and Aleksandrov maximum principle (see Theorem 3.4 below).

Now, for any $\varphi \in B^{2-1/p,p}(\partial\Omega)$, we can find a function $v \in W^{2,p}(\Omega)$ such that $\gamma_0 v = \varphi$. Hence we have the following existence and uniqueness theorem for the *non-homogeneous* Dirichlet problem:

Corollary 3.3. *For any $f \in L^p(\Omega)$ and any $\varphi \in B^{2-1/p,p}(\partial\Omega)$ with $1 < p < \infty$, there exists a unique solution $u \in W^{2,p}(\Omega)$ of the Dirichlet problem*

$$\begin{cases} \mathcal{L}_0 u = f & \text{in } \Omega, \\ \gamma_0 u = \varphi & \text{on } \partial\Omega. \end{cases} \quad (3.4)$$

If we associate with problem (3.4) a continuous linear operator

$$\mathcal{A}_0 = (\mathcal{L}_0, \gamma_0) : W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B^{2-1/p,p}(\partial\Omega),$$

then Corollary 3.3 asserts that the mapping \mathcal{A}_0 is an algebraic and topological *isomorphism*. In particular, we have

$$\text{ind } \mathcal{A}_0 = 0. \quad (3.5)$$

Step 2: If we let

$$B(\alpha)u = \sum_{i=1}^N b^i(x) \frac{\partial u}{\partial x_i} + (c(x) + \alpha)u, \quad \alpha \geq 0,$$

then it is clear that the operator

$$B(\alpha) : W^{2,p}(\Omega) \longrightarrow W^{1,p}(\Omega)$$

is continuous. However, it follows from an application of the Rellich and Kondrachov theorem (see [10, Section 7.12, Theorem 7.26]) that the injection $W^{1,p}(\Omega) \rightarrow L^p(\Omega)$ is compact. Hence we find that the mapping

$$B(\alpha) : W^{2,p}(\Omega) \longrightarrow L^p(\Omega)$$

is *compact*.

Therefore, we obtain that the mapping

$$\mathcal{A}(\alpha) = (\mathcal{L} + \alpha, \gamma_0) = \mathcal{A}_0 + (B(\alpha), 0) : W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B^{2-1/p,p}(\partial\Omega)$$

is a Fredholm operator with index *zero*, since we have, by assertion (3.5),

$$\text{ind } \mathcal{A}(\alpha) = \text{ind } \mathcal{A}_0 = 0.$$

Step 3: On the other hand, the *uniqueness* result in Theorem 3.1 follows from the Bakel'man and Aleksandrov maximum principle (cf. [3, Théorème 2]; [10, Section 9.1, Theorem 9.1]):

Theorem 3.4 (the weak maximum principle). *Let $\alpha \geq 0$. Assume that*

$$\begin{cases} u \in C(\overline{\Omega}) \cap W_{\text{loc}}^{2,N}(\Omega), \\ (\mathcal{L} + \alpha)u(x) \leq 0 \quad \text{for almost all } x \in \Omega. \end{cases}$$

Then it follows that

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+,$$

where

$$u^+(x) = \max\{u(x), 0\}, \quad x \in \overline{\Omega}.$$

By applying Theorem 3.4 to the functions $u(x)$ and $-u(x)$, we find that

$$\begin{cases} (\mathcal{L} + \alpha)u = 0 & \text{almost everywhere in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega \end{cases} \implies u = 0 \quad \text{in } \Omega.$$

This proves that the mapping

$$\mathcal{A}(\alpha) = (\mathcal{L} + \alpha, \gamma_0) : W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B^{2-1/p,p}(\partial\Omega)$$

is *injective* for $N < p < \infty$. Hence it is also *surjective* for $N < p < \infty$, since $\text{ind } \mathcal{A}(\alpha) = 0$.

Step 4: Summing up, we have proved that the mapping

$$\mathcal{A}(\alpha) = (\mathcal{L} + \alpha, \gamma_0) : W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \bigoplus B^{2-1/p,p}(\partial\Omega)$$

is an algebraic and topological *isomorphism* for $N < p < \infty$. \square

3.3 Proof of Theorem 1.1.

(I) First, we prove that the operator $L : C_0(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ is *closed*. To do this, we use the *a priori* estimate (3.2).

Let (u, v) be an arbitrary element of the space $C_0(\overline{\Omega}) \oplus C(\overline{\Omega})$ for which there exists a sequence $\{u_n\}$ in $D(L)$ such that

$$\begin{aligned} u_n &\longrightarrow u \quad \text{in } C_0(\overline{\Omega}), \\ \mathcal{L}u_n &\longrightarrow v \quad \text{in } C(\overline{\Omega}). \end{aligned}$$

Then, applying estimate (3.2) with $\alpha := 0$ to the sequence $\{u_n - u_m\}$ we obtain that

$$\|u_n - u_m\|_{W^{2,p}(\Omega)} \leq C \|\mathcal{L}u_n - \mathcal{L}u_m\|_{L^p(\Omega)}.$$

This implies that $\{u_n\}$ is a Cauchy sequence in $W^{2,p}(\Omega)$, since the injection $C(\overline{\Omega}) \rightarrow L^p(\Omega)$ is continuous. Hence it follows that

$$u \in W^{2,p}(\Omega), \quad (3.6)$$

and that

$$u_n \longrightarrow u \quad \text{in } W^{2,p}(\Omega). \quad (3.7)$$

Moreover, we have

$$\mathcal{L}u_n \longrightarrow \mathcal{L}u \quad \text{in } L^p(\Omega),$$

and so

$$\mathcal{L}u = v \in C(\overline{\Omega}). \quad (3.8)$$

On the other hand, by assertion (3.7) it follows that

$$\gamma_0 u = \lim_{n \rightarrow \infty} \gamma_0 u_n = 0 \quad \text{in } B^{2-1/p,p}(\partial\Omega).$$

This implies that

$$u \in W_0^{1,p}(\Omega). \quad (3.9)$$

Therefore, by combining assertions (3.6), (3.8) and (3.9) we have proved that

$$\begin{cases} u \in D(L), \\ \mathcal{L}u = v. \end{cases}$$

(II) Secondly, we prove that the equation $\mathcal{L}u = f$ has a unique solution $u \in D(L)$ for any $f \in C(\overline{\Omega})$.

By applying Theorem 3.1 with $\alpha := 0$, we obtain that the Dirichlet problem

$$\begin{cases} \mathcal{L}u = f & \text{almost everywhere in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in W^{2,p}(\Omega)$ for any $f \in C(\overline{\Omega})$ with $N < p < \infty$. In other words, for any $f \in C(\overline{\Omega})$ there exists a unique function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\mathcal{L}u = f \quad \text{in } \Omega.$$

Hence we have

$$\mathcal{L}u = f \in C(\overline{\Omega}).$$

This implies that

$$\begin{cases} u \in D(L), \\ \mathcal{L}u = f. \end{cases}$$

Moreover, the *continuity* of the inverse L^{-1} follows from an application of Banach's closed graph theorem.

(III) Finally, it remains to prove the *density* of the domain $D(L)$ in the space $C_0(\overline{\Omega})$. The proof is divided into four steps.

Step 1: First, we prove that, for each $\alpha > 0$ the equation $(\alpha + L)u = f$ has a unique solution $u \in D(L)$ for any $f \in C(\overline{\Omega})$.

It follows from an application of Theorem 3.1 that the Dirichlet problem

$$\begin{cases} (\alpha + \mathcal{L})u = f & \text{almost everywhere in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in W^{2,p}(\Omega)$ for any $f \in C(\overline{\Omega})$. More precisely, for any function $f \in C(\overline{\Omega})$ we can find a unique function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, with $N < p < \infty$, such that

$$(\alpha + \mathcal{L})u = f \quad \text{in } \Omega,$$

so that

$$\mathcal{L}u = f - \alpha u \in C(\overline{\Omega}).$$

This proves that

$$\begin{cases} u \in D(L), \\ (\alpha + L)u = f. \end{cases}$$

Step 2: Secondly, we prove that, for each $\alpha \geq 0$ the Green operator $G_\alpha^0 := (\alpha + L)^{-1}$ is *non-negative* on the space $C(\overline{\Omega})$:

$$f \in C(\overline{\Omega}), f(x) \geq 0 \quad \text{in } \Omega \implies u(x) = G_\alpha^0 f(x) \geq 0 \quad \text{in } \Omega.$$

Indeed, if we let

$$v(x) := -u(x) = -G_\alpha^0 f(x),$$

then it follows that

$$\begin{cases} (\mathcal{L} + \alpha)v = -f \leq 0 & \text{in } \Omega, \\ \gamma_0 v = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, applying Theorem 3.4 to the function $v(x)$ we obtain that

$$v(x) \leq 0 \quad \text{in } \Omega,$$

so that

$$u(x) = -v(x) \geq 0 \quad \text{in } \Omega.$$

Step 3: Thirdly, we prove that, for each $\alpha > 0$ the Green operator $G_\alpha^0 = (\alpha + L)^{-1}$ is *bounded* on the space $C(\overline{\Omega})$ with norm $1/\alpha$: $\|G_\alpha^0\| \leq 1/\alpha$.

Let $f(x)$ be an arbitrary function in $C(\overline{\Omega})$. If we let

$$u_\pm(x) = \pm \alpha G_\alpha^0 f(x) - \|f\|_{C(\overline{\Omega})} \in W^{2,p}(\Omega),$$

we have only to prove that

$$u_\pm(x) \leq 0 \quad \text{in } \Omega. \tag{3.10}$$

Indeed, it follows that

$$(\mathcal{L} + \alpha)u_\pm(x) = \pm \alpha f(x) - (c(x) + \alpha)\|f\|_{C(\overline{\Omega})}$$

$$\begin{aligned}
&= -\alpha(\|f\|_{C(\overline{\Omega})} \mp f(x)) - c(x)\|f\|_{C(\overline{\Omega})} \\
&\leq 0 \quad \text{in } \Omega.
\end{aligned}$$

Therefore, the desired assertion (3.10) follows from an application of Theorem 3.4 to the functions $u_{\pm}(x)$, since we have

$$u_{\pm} = -\|f\|_{C(\overline{\Omega})} \leq 0 \quad \text{on } \partial\Omega.$$

Step 4: Finally, we prove that the domain $D(L)$ is *dense* in $C_0(\overline{\Omega})$. More precisely, we prove that, for any $u \in C_0(\overline{\Omega})$,

$$\lim_{\alpha \rightarrow +\infty} \|\alpha G_{\alpha}^0 u - u\|_{C(\overline{\Omega})} = 0. \quad (3.11)$$

To do this, we introduce an extension \widetilde{G}_{α}^0 of the Green operator G_{α}^0 to the space $L^p(\Omega)$ for $N < p < \infty$:

$$\begin{array}{ccc}
L^p(\Omega) & \xrightarrow{\widetilde{G}_{\alpha}^0} & C_0(\overline{\Omega}) \\
\uparrow & & \downarrow \\
C(\overline{\Omega}) & \xrightarrow{G_{\alpha}^0} & D(L)
\end{array}$$

By applying Theorem 3.1, we obtain that the Dirichlet problem

$$\begin{cases} (\alpha + \mathcal{L})u = f & \text{almost everywhere in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in W^{2,p}(\Omega) \cap C_0(\overline{\Omega})$ for any $f \in L^p(\Omega)$. If we let

$$u = \widetilde{G}_{\alpha}^0 f,$$

then it is easy to verify that the operator \widetilde{G}_{α}^0 is an extension of G_{α}^0 to $L^p(\Omega)$. Moreover, just as in Steps 2 and 3, we can prove the following two assertions:

- (A) The operator $\widetilde{G}_{\alpha}^0 : L^p(\Omega) \rightarrow C_0(\overline{\Omega})$ is non-negative.
- (B) The operator $\widetilde{G}_{\alpha}^0 : L^{\infty}(\Omega) \rightarrow C_0(\overline{\Omega})$ is bounded with norm $1/\alpha$: $\|\widetilde{G}_{\alpha}^0\| \leq 1/\alpha$.

Since the space $C_0^2(\overline{\Omega}) := C_0(\overline{\Omega}) \cap C^2(\overline{\Omega})$ is dense in $C_0(\overline{\Omega})$, it suffices to prove assertion (3.11) for any $u \in C_0^2(\overline{\Omega})$.

First, since $a^{ij}, b^i \in L^{\infty}(\Omega)$ and $u \in C^2(\overline{\Omega})$, it follows that

$$\mathcal{L}u = - \sum_{i,j=1}^N a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b^i(x) \frac{\partial u}{\partial x_i} + c(x)u \in L^{\infty}(\Omega),$$

Hence, if we let

$$w = \alpha G_{\alpha}^0 u + \widetilde{G}_{\alpha}^0(\mathcal{L}u),$$

then we have

$$\begin{cases} w \in W^{2,p}(\Omega) \cap C_0(\overline{\Omega}), \\ (\mathcal{L} + \alpha)w = \alpha u + \mathcal{L}u = (\mathcal{L} + \alpha)u & \text{in } \Omega, \end{cases}$$

and so

$$\begin{cases} w - u \in W^{2,p}(\Omega) \cap C_0(\overline{\Omega}), \\ (\mathcal{L} + \alpha)(w - u) = 0 \quad \text{in } \Omega. \end{cases}$$

By applying Theorem 3.1 (or Theorem 3.4) to the function $w(x) - v(x)$, we obtain that $w - u = 0$ in Ω , that is,

$$u = w = \alpha G_\alpha^0 u + \widetilde{G}_\alpha^0(\mathcal{L}u).$$

Therefore, the desired assertion (3.11) for any $u \in C_0^2(\overline{\Omega})$ follows from an application of assertion (B). Indeed, we have, for all $\alpha > 0$,

$$\|u - \alpha G_\alpha^0 u\|_{C(\overline{\Omega})} = \|\widetilde{G}_\alpha^0(\mathcal{L}u)\|_{C(\overline{\Omega})} \leq \frac{1}{\alpha} \|\mathcal{L}u\|_{L^\infty(\Omega)}.$$

Now the proof of Theorem 1.1 is complete. \square

4. Proof of Part (i) of Theorem 1.2

This section is devoted to the proof of Part (i) of Theorem 1.2. By rescaling, we may assume that

$$|m(x)| < 1 \quad \text{on } \overline{\Omega}. \quad (4.1)$$

Our proof is divided into four steps.

Step 1: First, the *positivity* of the resolvent L^{-1} for problem (3.1) follows from an application of a variant of the strong maximum principle and of the boundary point lemma in the framework of Sobolev spaces (cf. [10, Section 9.1]):

Theorem 4.1 (the strong maximum principle). *Assume that*

$$\begin{cases} u \in C(\overline{\Omega}) \cap W_{\text{loc}}^{2,N}(\Omega), \\ \mathcal{L}u(x) \leq 0 \quad \text{almost everywhere in } \Omega, \\ d = \sup_\Omega u \geq 0. \end{cases}$$

If there is a point $x_0 \in \Omega$ such that $u(x_0) = d$, then it follows that

$$u(x) \equiv d \quad \text{for all } x \in \Omega.$$

Theorem 4.2 (the boundary point lemma). *Assume that*

$$\begin{cases} u \in C^1(\overline{\Omega}) \cap W_{\text{loc}}^{2,N}(\Omega), \\ \mathcal{L}u(x) \leq 0 \quad \text{for almost all } x \in \Omega, \end{cases}$$

and that there is a point $x'_0 \in \partial\Omega$ such that

$$\begin{cases} u(x'_0) \geq 0, \\ u(x'_0) > u(y) \quad \text{for all } y \in \Omega. \end{cases}$$

Then it follows that

$$\frac{\partial u}{\partial \mathbf{n}}(x'_0) < 0,$$

where $\mathbf{n} = (n_1, n_2, \dots, n_N)$ is the unit interior normal to $\partial\Omega$.

Indeed, by combining Theorem 1.1 and Theorems 4.1 and 4.2 we can obtain the following results for the resolvent L^{-1} :

Proposition 4.3. (i) The resolvent $L^{-1} : Y \rightarrow X$ is strongly positive; that is, $L^{-1}(P_Y \setminus \{0\}) \subset \text{Int}(P_X)$.

(ii) If $M : X \rightarrow Y$ is the multiplication operator by a function $m(x) \in C(\overline{\Omega})$, then the operator $L^{-1}M : X \rightarrow X$ is compact.

Step 2: The proof of Part (ii) of Theorem 1.2 may be carried out just as in Hess and Kato [12, Theorem 1], by using Proposition 4.3 and the Kreĭn and Rutman theorem (Theorem 2.4).

Step 2-1: First, we prove the following fundamental lemma (see [12, Lemma 2]):

Lemma 4.4. If $m(x) \in C(\overline{\Omega})$ is positive somewhere in Ω , then there exist a constant $\alpha_0 > 0$ and a function $w_0 \in P_Y \setminus \{0\}$ which satisfy the condition

$$\alpha_0 K_{\alpha_0} w_0 - w_0 \in P_Y, \quad (4.2)$$

where

$$K_{\alpha_0} := (\alpha_0 + L)^{-1}(M + 1).$$

Proof. The proof of Lemma 4.4 is essentially the same as in Hess and Kato [12, Lemma 2] if we make use of Proposition 4.3 and Theorem 3.4 (the weak maximum principle).

(i) Now let x_0 be a point of Ω such that

$$m(x_0) > 0.$$

Then, since $m(x) \in C(\overline{\Omega})$, we can find constants $\rho > 0$ and $\delta > 0$ such that

$$m(x) \geq \delta, \quad x \in B(x_0, \rho), \quad (4.3)$$

where $B(x_0, \rho)$ is the open ball of radius ρ about x_0 .

We consider the linear Dirichlet eigenvalue problem

$$\begin{cases} \mathcal{L}\tilde{w} = \gamma\tilde{w} & \text{in } B(x_0, \rho), \\ \tilde{w} = 0 & \text{on } \partial B(x_0, \rho). \end{cases} \quad (4.4)$$

By applying Proposition 4.3 and Theorem 2.4 to our situation, we obtain that problem (4.3) has a positive principal eigenvalue γ_1 , with associated positive eigenfunction $\tilde{w}_1 \in W^{2,p}(B(x_0, \rho)) \cap W_0^{1,p}(B(x_0, \rho))$ for $p > N$.

If we let

$$\alpha_0 := \frac{\gamma_1}{\delta},$$

then we have

$$\begin{cases} (\mathcal{L} + \alpha_0)\tilde{w}_1 = \alpha_0(\delta + 1)\tilde{w}_1 & \text{in } B(x_0, \rho), \\ \tilde{w}_1 = 0 & \text{on } \partial B(x_0, \rho). \end{cases}$$

(ii) We can define a function $w_0(x) \in Y = C(\overline{\Omega})$ by the formula

$$w_0(x) := \begin{cases} \tilde{w}_1(x) & \text{in } B(x_0, \rho), \\ 0 & \text{in } \overline{\Omega} \setminus B(x_0, \rho). \end{cases} \quad (4.5)$$

It remains to verify that the function $w_0(x)$ satisfies condition (4.2). To do this, we let

$$v = \alpha_0(\alpha_0 + L)^{-1}((\delta + 1)w_0).$$

Then we have, by Proposition 4.3,

$$v = \alpha_0(\alpha_0 + L)^{-1}((\delta + 1)w_0) \in \text{Int}(P_X), \quad (4.6)$$

and hence

$$v(x) \geq 0 = w_0(x) \quad \text{in } \overline{\Omega} \setminus B(x_0, \rho). \quad (4.7)$$

Moreover, it should be noticed that

$$\begin{cases} (\mathcal{L} + \alpha_0)w_0 = \alpha_0(\delta + 1)w_0 & \text{in } B(x_0, \rho), \\ w_0 = 0 & \text{on } \partial B(x_0, \rho), \end{cases}$$

and that, by assertion (4.6),

$$\begin{cases} (\mathcal{L} + \alpha_0)v = \alpha_0(\delta + 1)w_0 & \text{in } B(x_0, \rho), \\ v > 0 & \text{on } \partial B(x_0, \rho). \end{cases}$$

This implies that

$$\begin{cases} (\mathcal{L} + \alpha_0)(w_0 - v) = 0 & \text{in } B(x_0, \rho), \\ w_0 - v < 0 & \text{on } \partial B(x_0, \rho). \end{cases}$$

Therefore, applying Theorem 3.4 to the function $w_0(x) - v(x)$ we obtain that

$$\sup_{B(x_0, \rho)} (w_0 - v) \leq \sup_{\partial B(x_0, \rho)} (w_0 - v)^+ = 0,$$

so that

$$v(x) \geq w_0(x) \quad \text{in } B(x_0, \rho). \quad (4.8)$$

By combining assertions (4.7) and (4.8), we have proved that

$$w_0 \leq \alpha_0(\alpha_0 + L)^{-1}((\delta + 1)w_0) \quad \text{in } \Omega. \quad (4.9)$$

On the other hand, since we have, by condition (4.3) and definition (4.5),

$$\alpha_0(\delta + 1)w_0(x) \leq \alpha_0(m(x) + 1)w_0(x) \quad \text{in } \Omega,$$

it follows from the positivity of the Green operator $(\alpha_0 + L)^{-1}$ that

$$\begin{aligned} \alpha_0(\alpha_0 + L)^{-1}((\delta + 1)w_0) &\leq \alpha_0(\alpha_0 + L)^{-1}((m(x) + 1)w_0) \\ &= \alpha_0 K_{\alpha_0} w_0 \quad \text{in } \Omega. \end{aligned} \quad (4.10)$$

Finally, the desired assertion (4.2) follows by combining assertions (4.9) and (4.10). \square

Step 2-2: The next lemma, essentially due to Hess and Kato [12, Lemma 1], asserts that condition (4.2) implies the existence of a positive eigenvalue of the equation

$$Lu = \lambda Mu. \quad (1.3)$$

Lemma 4.5. *If a constant $\alpha_0 > 0$ and a function $w_0 \in P_Y \setminus \{0\}$ satisfy condition (4.2), then we can find a constant $\lambda \in (0, \alpha_0]$ and a function $u \in P_X \setminus \{0\}$ such that equation (1.3) holds.*

Moreover, if we have

$$\alpha_0 K_{\alpha_0} w_0 - w_0 \in \text{Int}(P_Y), \quad (4.11)$$

then it follows that $0 < \lambda < \alpha_0$.

Proof. (i) Since $K_{\alpha_0} : Y \rightarrow Y$ is strongly positive and compact, it follows from an application of the Kreĭn and Rutman theorem (Theorem 2.4) that the spectral radius

$$\text{spr}(K_{\alpha_0}) := \lim_{n \rightarrow \infty} \|(K_{\alpha_0})^n\|^{1/n}$$

is an eigenvalue of K_{α_0} having a positive eigenfunction w_1 , normalized as $\|w_1\|_Y = 1$,

$$\begin{cases} K_{\alpha_0} w_1 = \text{spr}(K_{\alpha_0}) w_1, \\ w_1 > 0, \quad \|w_1\|_Y = 1. \end{cases} \quad (4.12)$$

If we let

$$\alpha_1 := \frac{1}{\text{spr}(K_{\alpha_0})},$$

then we have, by formula (4.12),

$$\alpha_1 \leq \alpha_0. \quad (4.13)$$

Indeed, since $\text{spr}(K_{\alpha_0})$ is also an eigenvalue of the adjoint operator $(K_{\alpha_0})^*$ with a positive eigenfunction $w_1^* \in Y^*$ and since $K_{\alpha_0} w_0 \in \text{Int}(P_Y)$, it follows that

$$\langle w_1^*, K_{\alpha_0} w_0 \rangle > 0.$$

This proves that

$$0 < \langle w_1^*, K_{\alpha_0} w_0 \rangle = \langle ((K_{\alpha_0})^* w_1^*, w_0 \rangle = \frac{1}{\alpha_1} \langle w_1^*, w_0 \rangle,$$

so that

$$\langle w_1^*, w_0 \rangle > 0. \quad (4.14)$$

Hence, if condition (4.2) is satisfied, then we have

$$\begin{aligned} 0 < \langle w_1^*, w_0 \rangle &\leq \alpha_0 \langle w_1^*, K_{\alpha_0} w_0 \rangle \\ &= \alpha_0 \langle ((K_{\alpha_0})^* w_1^*, w_0 \rangle \\ &= \frac{\alpha_0}{\alpha_1} \langle w_1^*, w_0 \rangle. \end{aligned} \quad (4.15)$$

Therefore, the desired assertion (4.13) follows by combining inequalities (4.14) and (4.15).

Moreover, it should be emphasized that if condition (4.11) is satisfied, then we have

$$0 < \langle w_1^*, w_0 \rangle < \alpha_0 \langle w_1^*, K_{\alpha_0} w_0 \rangle$$

$$\begin{aligned}
&= \alpha_0 \langle (K_{\alpha_0})^* w_1^*, w_0 \rangle \\
&= \frac{\alpha_0}{\alpha_1} \langle w_1^*, w_0 \rangle,
\end{aligned}$$

and so

$$\alpha_1 < \alpha_0. \quad (4.16)$$

(ii) Since we have, by formula (4.12),

$$(L + \alpha_0)^{-1}(M + 1)w_1 = \frac{1}{\alpha_1} w_1,$$

it follows that

$$\begin{aligned}
(L + \alpha_1)w_1 &= (L + \alpha_0)w_1 + (\alpha_1 - \alpha_0)w_1 \\
&= \alpha_1(M + 1)w_1 - (\alpha_0 - \alpha_1)w_1.
\end{aligned}$$

By the positivity of $(L + \alpha_1)^{-1}$, this proves that

$$\begin{aligned}
w_1 &= \alpha_1(L + \alpha_1)^{-1}(M + 1)w_1 - (\alpha_0 - \alpha_1)(L + \alpha_1)^{-1}w_1 \\
&= \alpha_1 K_{\alpha_1} w_1 - (\alpha_0 - \alpha_1)(L + \alpha_1)^{-1}w_1 \\
&\leq \alpha_1 K_{\alpha_1} w_1,
\end{aligned}$$

that is,

$$\alpha_1 K_{\alpha_1} w_1 - w_1 \in P_Y.$$

(iii) Similarly, if we let

$$\alpha_2 := \frac{1}{\text{spr}(K_{\alpha_1})},$$

then we can find a positive eigenfunction w_2 of K_{α_1} , normalized as $\|w_2\|_Y = 1$,

$$\begin{cases} K_{\alpha_1} w_2 = \frac{1}{\alpha_2} w_2, \\ w_2 > 0, \quad \|w_2\|_Y = 1, \end{cases} \quad (4.17)$$

and

$$\alpha_2 \leq \alpha_1.$$

Moreover, we have, by formula (4.17),

$$(L + \alpha_2)w_2 = \alpha_2(M + 1)w_2 - (\alpha_1 - \alpha_2)w_2,$$

and so

$$\begin{aligned}
w_2 &= \alpha_2(L + \alpha_2)^{-1}(M + 1)w_2 - (\alpha_1 - \alpha_2)(L + \alpha_2)^{-1}w_2 \\
&= \alpha_2 K_{\alpha_2} w_2 - (\alpha_1 - \alpha_2)(L + \alpha_2)^{-1}w_2 \\
&\leq \alpha_2 K_{\alpha_2} w_2.
\end{aligned}$$

This proves that

$$\alpha_2 K_{\alpha_2} w_2 - w_2 \in P_Y.$$

Repeating this process, we can construct a sequence $\{\alpha_j\} \subset \mathbf{R}^+$ and a sequence $\{w_j\} \subset P_Y$ which satisfy the conditions

$$0 < \dots \leq \alpha_2 \leq \alpha_1 \leq \alpha_0, \quad (4.18a)$$

$$w_j > 0, \quad \|w_j\|_Y = 1, \quad (4.18b)$$

and

$$w_j = \alpha_j K_{\alpha_j} w_j - (\alpha_{j-1} - \alpha_j)(L + \alpha_j)^{-1} w_j. \quad (4.19)$$

(iv) By assertion (4.18a), it follows that the sequence $\{\alpha_j\}$ converges to some $\lambda \in [0, \alpha_0]$. Then we have

$$(L + \alpha_j)^{-1} \longrightarrow (L + \lambda)^{-1} \quad \text{in } B(Y, Y), \quad (4.20)$$

and so

$$K_{\alpha_j} = (L + \alpha_j)^{-1}(M + 1) \longrightarrow K_\lambda = (L + \lambda)^{-1}(M + 1) \quad \text{in } B(Y, Y). \quad (4.21)$$

Therefore, combining assertions (4.20) and (4.18b) we obtain from formula (4.19) that

$$w_j - \alpha_j K_{\alpha_j} w_j = -(\alpha_{j-1} - \alpha_j)(L + \alpha_j)^{-1} w_j \longrightarrow 0 \quad \text{in } Y. \quad (4.22)$$

(v) We show that the sequence $\{w_j\}$ is *relatively compact* in Y .

Indeed, since $K_\lambda : Y \rightarrow Y$ is compact and since $\|w_j\|_Y = 1$, we can find a subsequence $\{w_{j'}\}$ such that $\{K_\lambda w_{j'}\}$ is a Cauchy sequence in Y . Then we have, by assertions (4.22) and (4.21),

$$\begin{aligned} w_{j'} - w_{k'} &= \left(w_{j'} - \alpha_{j'} K_{\alpha_{j'}} w_{j'} \right) + \alpha_{j'} K_{\alpha_{j'}} w_{j'} \\ &\quad - \alpha_{k'} K_{\alpha_{k'}} w_{k'} + \left(\alpha_{k'} K_{\alpha_{k'}} w_{k'} - w_{k'} \right) \\ &= \left(w_{j'} - \alpha_{j'} K_{\alpha_{j'}} w_{j'} \right) + \left(\alpha_{k'} K_{\alpha_{k'}} w_{k'} - w_{k'} \right) \\ &\quad + (\alpha_{j'} - \lambda) K_{\alpha_{j'}} w_{j'} + \lambda \left(K_{\alpha_{j'}} - K_\lambda \right) w_{j'} \\ &\quad - (\alpha_{k'} - \lambda) K_{\alpha_{k'}} w_{k'} - \lambda \left(K_{\alpha_{k'}} - K_\lambda \right) w_{k'} \\ &\quad + \lambda (K_\lambda w_{j'} - K_\lambda w_{k'}) \longrightarrow 0, \end{aligned}$$

as $j', k' \rightarrow \infty$.

(vi) By Step (v), we can find a subsequence $\{w_{j'}\}$ which converges to some function $u \in P_Y$ with $\|u\|_Y = 1$. Then it follows that, as $j' \rightarrow \infty$,

$$\begin{aligned} &w_{j'} - \alpha_{j'} K_{\alpha_{j'}} w_{j'} - (u - \lambda K_\lambda u) \\ &= (w_{j'} - u) - \left(\alpha_{j'} K_{\alpha_{j'}} w_{j'} - \lambda K_\lambda u \right) \\ &= (w_{j'} - u) - (\alpha_{j'} - \lambda) K_{\alpha_{j'}} w_{j'} - \lambda \left(K_{\alpha_{j'}} - K_\lambda \right) w_{j'} \\ &\quad - \lambda K_\lambda (w_{j'} - u) \longrightarrow 0. \end{aligned} \quad (4.23)$$

Therefore, combining assertions (4.22) and (4.23) we obtain that

$$u - \lambda K_\lambda u = 0,$$

so that

$$\begin{cases} u = \lambda(L + \lambda)^{-1}(M + 1)u \in P_X, \\ Lu = \lambda Mu. \end{cases}$$

Here it should be noticed that $\lambda > 0$, since $\|u\|_Y = 1$.

Summing up, we have proved that there exist a constant $\lambda \in (0, \alpha_0]$ and a function $u \in P_X \setminus \{0\}$ such that equation (1.3) holds.

(vii) Finally, if inequality (4.11) holds, then we have, by inequality (4.16),

$$0 < \lambda \leq \alpha_1 < \alpha_0.$$

Now the proof of Lemma 4.6 is complete. \square

Step 2-3: By Lemmas 4.4 and 4.5, we find that the set

$$\Lambda(m) = \{\lambda > 0 : \lambda \text{ is an eigenvalue of equation (1.3) with a positive eigenfunction}\}$$

is *non-empty*. We let

$$\lambda_1(m) = \inf \Lambda(m).$$

Then we have the following:

Claim 4.6. *The infimum in $\Lambda(m)$ is attained. Namely, $\lambda_1(m)$ is a positive eigenvalue of equation (1.3) with a positive eigenfunction.*

Proof. Indeed, let $\{\lambda_j\}$ be a sequence in $\Lambda(m)$ such that

$$\lambda_j \longrightarrow \lambda_1(m),$$

and that

$$(L^{-1} M)u_j = \frac{1}{\lambda_j} u_j,$$

with

$$u_j \in P_X \setminus \{0\}.$$

Without loss of generality, we may assume that

$$\|u_j\|_X = 1.$$

Then, since the operator $L^{-1} M : X \rightarrow X$ is compact, we can find a subsequence $\{u_{j'}\}$ and an element $v \in X$ such that

$$\frac{u_{j'}}{\lambda_{j'}} = (L^{-1} M)u_{j'} \longrightarrow v \quad \text{in } X.$$

Hence we have

$$u_{j'} = \lambda_{j'} \left(\frac{u_{j'}}{\lambda_{j'}} \right) \longrightarrow \lambda_1(m)v \quad \text{in } X,$$

and also

$$\lambda_1(m) \|v\|_X = \lim_{j' \rightarrow \infty} \|u_{j'}\|_X = 1.$$

Therefore, we obtain that

$$\lambda_1(m) > 0,$$

and that

$$\begin{aligned} v &\in P_X \setminus \{0\}, \\ (L^{-1}M)v &= \frac{1}{\lambda_1(m)}v, \end{aligned}$$

or equivalently,

$$\begin{aligned} v &\in P_X \setminus \{0\}, \\ Lv &= \lambda_1(m)Mv. \end{aligned}$$

This proves that $\lambda_1(m) \in \Lambda(m)$. \square

Step 3: We look at the equation

$$Lv = \hat{\lambda}Mv, \quad \hat{\lambda} \in \mathbf{C}, \quad (4.24)$$

which is obtained by the *complexification* of equation (1.3).

Step 3-1: The next lemma, essentially due to Hess and Kato [12, Lemma 3], plays a fundamental role in the proof of Part (i) of Theorem 1.2:

Lemma 4.7. *Let $\hat{\lambda}$ be an eigenvalue of equation (4.24), with $\operatorname{Re} \hat{\lambda} > 0$, and let $v \in D(L)$ be its associated eigenfunction. Then we have*

$$|v| \leq \operatorname{Re} \hat{\lambda} K_{\operatorname{Re} \hat{\lambda}} |v|. \quad (4.25)$$

Proof. (i) If $u \in W^{2,p}(\Omega)$ with $p > N$ and if $\varepsilon > 0$, we define a function $u_\varepsilon \in W^{2,p}(\Omega)$ by the formula

$$u_\varepsilon(x) = \sqrt{|u(x)|^2 + \varepsilon^2}.$$

Moreover, for complex-valued functions $u \in W^{2,p}(\Omega)$ we define the differential expression $\mathcal{L}'u$ as follows:

$$\begin{aligned} \mathcal{L}'u &:= \mathcal{L}u - c(x)u \\ &= - \sum_{i,j=1}^N a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b^i(x) \frac{\partial u}{\partial x_i}. \end{aligned}$$

Then, just as in the proof of Kato [14, Lemma 3] we obtain that

$$\mathcal{L}'u_\varepsilon \leq \operatorname{Re} \left(\frac{\overline{u}}{u_\varepsilon} \cdot \mathcal{L}'u \right). \quad (4.26)$$

It should be noticed that inequality (4.26) asserts the difference of the both sides is a non-negative *distribution*. To prove inequality (4.26), it suffices to note the inequality

$$\sum_{i,j=1}^N a^{ij}(x) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} \leq \sum_{i,j=1}^N a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

(ii) If $\hat{\lambda}$ is an eigenvalue of equation (4.24) and if $v \in D(L)$ is its associated eigenfunction, then it follows that $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $p > N$. Hence, applying inequality (4.26) to the function v we obtain that

$$\begin{aligned}
\mathcal{L}v_\varepsilon &= \mathcal{L}'v_\varepsilon + c(x)v_\varepsilon \\
&\leq \operatorname{Re} \left[\frac{\bar{v}}{v_\varepsilon} \cdot \mathcal{L}'v \right] + c(x)v_\varepsilon \\
&= \operatorname{Re} \left[\frac{\bar{v}}{v_\varepsilon} (\mathcal{L}v - c(x)v) + c(x)v_\varepsilon \right] \\
&= \operatorname{Re} \left[\frac{\bar{v}}{v_\varepsilon} \cdot \mathcal{L}v + c(x) \left(v_\varepsilon - \frac{|v|^2}{v_\varepsilon} \right) \right] \\
&= \operatorname{Re} \left[\frac{\bar{v}}{v_\varepsilon} \cdot \hat{\lambda} m(x)v \right] + c(x) \left(v_\varepsilon - \frac{|v|^2}{v_\varepsilon} \right) \\
&= (\operatorname{Re} \hat{\lambda}) m(x) \left(\frac{|v|^2}{v_\varepsilon} \right) + c(x) \left(v_\varepsilon - \frac{|v|^2}{v_\varepsilon} \right),
\end{aligned}$$

so that

$$(\mathcal{L} + \operatorname{Re} \hat{\lambda}) v_\varepsilon \leq \operatorname{Re} \hat{\lambda} \left(m(x) \cdot \frac{|v|^2}{v_\varepsilon} + v_\varepsilon \right) + c(x) \left(v_\varepsilon - \frac{|v|^2}{v_\varepsilon} \right). \quad (4.27)$$

However, we have

$$v_\varepsilon - \varepsilon \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

and also, by inequality (4.27),

$$\begin{aligned}
(\mathcal{L} + \operatorname{Re} \hat{\lambda}) (v_\varepsilon - \varepsilon) &= (\mathcal{L} + \operatorname{Re} \hat{\lambda}) v_\varepsilon - c(x)\varepsilon - (\operatorname{Re} \hat{\lambda}) \varepsilon \\
&\leq \operatorname{Re} \hat{\lambda} \left(m(x) \cdot \frac{|v|^2}{v_\varepsilon} + v_\varepsilon \right) + c(x) \left(v_\varepsilon - \frac{|v|^2}{v_\varepsilon} \right) \\
&\quad - \varepsilon (c(x) + \operatorname{Re} \hat{\lambda}),
\end{aligned} \quad (4.28)$$

where

$$m(x) \cdot \frac{|v|^2}{v_\varepsilon} + v_\varepsilon \in C(\overline{\Omega}), \quad c(x) \in L^\infty(\Omega).$$

(iii) Recall (see assertion (A) in Subsection 3.3) that the Green operator

$$(\operatorname{Re} \hat{\lambda} + \mathcal{L})^{-1} : L^p(\Omega) \longrightarrow C_0(\overline{\Omega})$$

is non-negative. Thus, applying the Green operator to the both sides of inequality (4.28) we obtain that

$$\begin{aligned}
v_\varepsilon - \varepsilon &\leq \operatorname{Re} \hat{\lambda} (\operatorname{Re} \hat{\lambda} + \mathcal{L})^{-1} \left(m(x) \cdot \frac{|v|^2}{v_\varepsilon} + v_\varepsilon \right) \\
&\quad + (\operatorname{Re} \hat{\lambda} + \mathcal{L})^{-1} \left(c(x) \left(v_\varepsilon - \frac{|v|^2}{v_\varepsilon} \right) - \varepsilon (c(x) + \operatorname{Re} \hat{\lambda}) \right).
\end{aligned} \quad (4.29)$$

Therefore, the desired inequality (4.25) follows by passing to the limit $\varepsilon \downarrow 0$ in inequality (4.29), since we have

$$\begin{aligned} v_\varepsilon &\longrightarrow |v| \quad \text{in } C(\overline{\Omega}), \\ \frac{|v|^2}{v_\varepsilon} &\longrightarrow |v| \quad \text{in } C(\overline{\Omega}). \end{aligned}$$

The proof of Lemma 4.7 is complete. \square

Step 3-2: By combining Lemmas 4.5 and 4.7, we obtain that if $\hat{\lambda}$ is an eigenvalue of equation (4.22) and if $\operatorname{Re} \hat{\lambda} > 0$, then we have

$$\operatorname{Re} \hat{\lambda} \geq \lambda_1(m).$$

Indeed, if v is an eigenfunction of equation (4.24), then we have, by Lemma 4.7,

$$|v| \leq \operatorname{Re} \hat{\lambda} K_{\operatorname{Re} \hat{\lambda}} |v|.$$

Therefore, by applying Lemma 4.5 with

$$\alpha_0 := \operatorname{Re} \hat{\lambda}, \quad w_0 := |v|,$$

we can find a constant $0 < \lambda \leq \operatorname{Re} \hat{\lambda}$ and a function $u \in P_X \setminus \{0\}$ such that

$$Lu = \lambda Mu.$$

This implies that $\lambda \in \Lambda(m)$, so that

$$\lambda_1(m) \leq \lambda \leq \operatorname{Re} \hat{\lambda}.$$

The proof of Part (i) of Theorem 1.2 is now complete.

5. Proof of Part (ii) of Theorem 1.2

This section is devoted to the proof of Part (ii) of Theorem 1.2. The proof of Part (ii) of Theorem 1.2 can be accomplished in a series of lemmas, just as in Hess and Kato [12].

Step 1: First, we let

$$\bar{t} := \max_{x \in \overline{\Omega}} m^+(x),$$

where

$$m^+(x) = \max\{m(x), 0\},$$

and consider the eigenvalue problem

$$Lu = \lambda(M - t)u, \quad t \in I := (-\infty, \bar{t}). \quad (5.1)_t$$

Since the function $m(x) - t$ is positive somewhere in Ω for $t \in I$, applying Part (i) of Theorem 1.2 we obtain that equation $(5.1)_t$ admits a unique positive eigenvalue $\lambda_t := \lambda_1(m - t)$ with a positive eigenfunction $u_t \in \operatorname{Int}(P_Y)$.

Step 1-1: We show that the function $\lambda_t = \lambda_1(m - t)$ is *strictly monotone increasing* in $t \in I = (-\infty, \bar{t})$. Indeed, it suffices to note the following comparison result for indefinite weight functions (see [12, Proposition 1]):

Proposition 5.1. *Let $m_1(x)$ and $m_2(x)$ be two weight functions in $Y := C(\overline{\Omega})$ such that $m_2 < m_1$. Assume that $m_2(x_0) > 0$ for some point $x_0 \in \Omega$. Then it follows that $0 < \lambda_1(m_1) < \lambda_1(m_2)$.*

Proof. By rescaling, we may assume that

$$|m_1(x)| < 1, \quad |m_2(x)| < 1 \quad \text{on } \overline{\Omega}.$$

Now let $u_2(x)$ be a positive eigenfunction corresponding to the eigenvalue $\lambda_1(m_2)$:

$$\begin{aligned} Lu_2 &= \lambda_1(m_2)M_2u_2, \\ u_2 &> 0. \end{aligned}$$

Then we have

$$(L + \lambda_1(m_2))u_2 = \lambda_1(m_2)(M_2 + 1)u_2,$$

and so

$$u_2 = \lambda_1(m_2)(L + \lambda_1(m_2))^{-1}((M_2 + 1)u_2), \quad (5.2)$$

since $\lambda_1(m_2) > 0$. However, by assertion (3.6) with $\alpha := \lambda_1(m_2)$ we find that the operator

$$(L + \lambda_1(m_2))^{-1}(M_2 + 1) : Y \longrightarrow Y$$

is strongly positive. Thus it follows from formula (5.2) that

$$u_2 \in \text{Int}(P_Y),$$

so that

$$(m_2(x) + 1)u_2 < (m_1(x) + 1)u_2.$$

Hence we have, by the strong positivity of $(L + \lambda_1(m_2))^{-1}$,

$$\begin{aligned} \lambda_1(m_2)K_{\lambda_1(m_2)}u_2 &:= \lambda_1(m_2)(L + \lambda_1(m_2))^{-1}((M_1 + 1)u_2) \\ &\gg \lambda_1(m_2)(L + \lambda_1(m_2))^{-1}((M_2 + 1)u_2) \\ &= u_2, \end{aligned}$$

and so

$$\lambda_1(m_2)K_{\lambda_1(m_2)}u_2 - u_2 \in \text{Int}(P_Y).$$

Therefore, applying Lemma 4.6 with

$$m(x) := m_1(x), \quad \alpha_0 := \lambda_1(m_2), \quad w_0 := u_2,$$

we can find an eigenvalue $\lambda \in (0, \lambda_1(m_2))$ and an eigenfunction $u \in P_X \setminus \{0\}$ such that

$$Lu = \lambda M_1 u.$$

This proves that $\lambda \in \Lambda(m_1)$, so that

$$\lambda_1(m_1) \leq \lambda < \lambda_1(m_2).$$

The proof of Proposition 5.1 is complete. \square

Step 1-2: Moreover, the next lemma asserts that the function $\lambda_t = \lambda_1(m - t)$ is *continuous* in $t \in I = (-\infty, \bar{t})$ (see [12, Lemma 4]):

Lemma 5.2. *The reciprocal function $\mu_1(m - t) := 1/\lambda_t$ is continuous in $t \in I$.*

Proof. First, by Proposition 5.1 it follows that the function $\mu_1(m - t)$ is continuous except at at most countably many points, since it is strictly monotone decreasing in $t \in I$.

Now assume, to the contrary, that $t_0 \in I$ is a point of discontinuity of $\mu_1(m - t)$ (see Figure 5.1). Let $\{t_n\}$ be a sequence in I such that $t_n \uparrow t_0$, and let

$$\bar{\mu} = \lim_{n \rightarrow \infty} \mu_1(m - t_n).$$

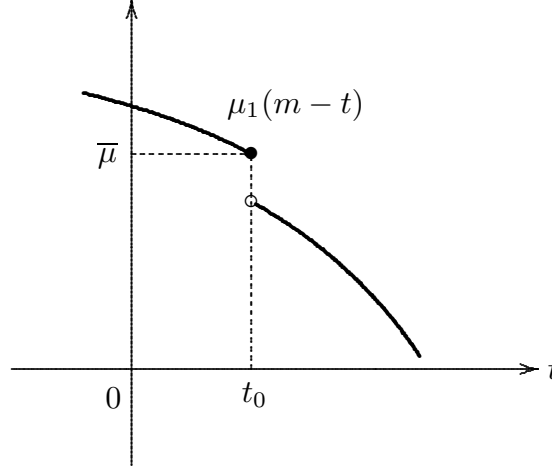


Figure 5.1

Then there exists a sequence $\{u_n\}$ of positive eigenfunctions associated with $\mu_1(m - t_n)$:

$$\begin{cases} u_n > 0, & \|u_n\|_Y = 1, \\ L^{-1}(M - t_n)u_n = \mu_1(m - t_n)u_n. \end{cases}$$

Since the operator $L^{-1} : Y \rightarrow Y$ is compact, we can find a subsequence $\{u_{n'}\}$ and a positive function u in Y such that

$$\begin{cases} u_{n'} \rightarrow u, \\ \|u\|_Y = 1, \\ L^{-1}(M - t_0)u = \bar{\mu}u. \end{cases}$$

It should be noticed that $\bar{\mu}$ is an isolated eigenvalue of $L^{-1}(M - t_0)$, since $L^{-1}(M - t_0) : Y \rightarrow Y$ is compact.

If $\kappa \in \mathbf{C}$, we let

$$\hat{S}(\kappa) := \hat{L}^{-1}(\hat{M} - \kappa)$$

be a holomorphic family of compact mappings in the complexification \hat{Y} of Y . By applying analytic perturbation theory (see Kato [15]), we can find a *continuous branch* $\hat{\nu}(\kappa)$ of eigenvalues, for κ in a complex neighborhood of t_0 , such that

$$\begin{cases} \hat{\nu}(t_0) = \bar{\mu}, \\ \hat{L}^{-1}(\hat{M} - \kappa)v = \hat{\nu}(\kappa)v. \end{cases} \quad (5.3)$$

Hence, by the monotonicity of $\mu_1(m-t)$ and its discontinuity at $t = t_0$ it follows that

$$\operatorname{Re} \hat{\nu}(t) > \mu_1(m-t), \quad t_0 < t < t_0 + \varepsilon, \quad (5.4)$$

for $\varepsilon > 0$ sufficiently small.

On the other hand, since we have, by formula (5.3),

$$\hat{L}v = \frac{1}{\hat{\nu}(t)} \left(\hat{M} - t \right) v, \quad t_0 < t < t_0 + \varepsilon,$$

applying Lemma 4.7 with

$$M := M - t, \quad \hat{\lambda} := \frac{1}{\hat{\nu}(t)},$$

we obtain that

$$|v| \leq \operatorname{Re} \left(\frac{1}{\hat{\nu}(t)} \right) \left(L + \operatorname{Re} \left(\frac{1}{\hat{\nu}(t)} \right) \right)^{-1} (M - t + 1) |v|.$$

Moreover, by applying Lemma 4.5 with

$$w_0 := |v|, \quad \alpha_0 := \operatorname{Re} \left(\frac{1}{\hat{\nu}(t)} \right), \quad M := M - t,$$

we can find a constant $\gamma_t > 0$ and a function $u_t > 0$ such that

$$\begin{cases} \gamma_t \leq \operatorname{Re} \left(\frac{1}{\hat{\nu}(t)} \right), \\ Lu_t = \gamma_t(M-t)u_t, \quad t_0 < t < t_0 + \varepsilon. \end{cases}$$

This proves that

$$\lambda_1(m-t) \leq \operatorname{Re} \left(\frac{1}{\hat{\nu}(t)} \right), \quad t_0 < t < t_0 + \varepsilon,$$

so that

$$\frac{1}{\operatorname{Re} \left(\frac{1}{\hat{\nu}(t)} \right)} \leq \frac{1}{\lambda_1(m-t)} = \mu_1(m-t), \quad t_0 < t < t_0 + \varepsilon. \quad (5.5)$$

However, it should be noticed that

$$\frac{1}{\operatorname{Re} \left(\frac{1}{z} \right)} \geq \operatorname{Re}(z)$$

for any $z \in \mathbf{C}$ with $\operatorname{Re}(z) > 0$. Hence we have, by assertion (5.5),

$$\operatorname{Re}(\hat{\nu}(t)) \leq \frac{1}{\operatorname{Re} \left(\frac{1}{\hat{\nu}(t)} \right)} \leq \mu_1(m-t), \quad t_0 < t < t_0 + \varepsilon.$$

This contradicts assertion (5.4). \square

Step 2: Now we prove that (cf. [12, Lemma 5])

$$\lim_{t \uparrow \bar{t}} \lambda_t = +\infty. \quad (5.6)$$

Indeed, assume, to the contrary, that

$$\bar{\lambda} := \lim_{t \uparrow \bar{t}} \lambda_t < +\infty.$$

Then we can find a function $u_t \in P_X \setminus \{0\}$ such that

$$Lu_t = \lambda_t(M - t)u_t, \quad t \in I. \quad (5.7)$$

By passing to the limit $t \uparrow \bar{t}$ (for a subsequence) in formula (5.7) just as in the proof of Lemma 5.2, we obtain that

$$\begin{cases} Lu_{\bar{t}} = \bar{\lambda}(M - \bar{t})u_{\bar{t}}, \\ u_{\bar{t}} \in P_X \setminus \{0\}. \end{cases}$$

However, since $m(x) - \bar{t} \leq 0$ on $\bar{\Omega}$ and L^{-1} is strongly positive, it follows that

$$0 < u_{\bar{t}} = \frac{1}{\bar{\lambda}} L^{-1}(M - \bar{t})u_{\bar{t}} \leq 0.$$

This contradiction proves assertion (5.6).

Summing up, we have proved that the function $\lambda_t := \lambda_1(m - t)$ is strictly monotone increasing and continuous in $t \in I := (-\infty, \bar{t})$ (see Figure 5.2), and

$$\lim_{t \uparrow \bar{t}} \lambda_t = +\infty.$$

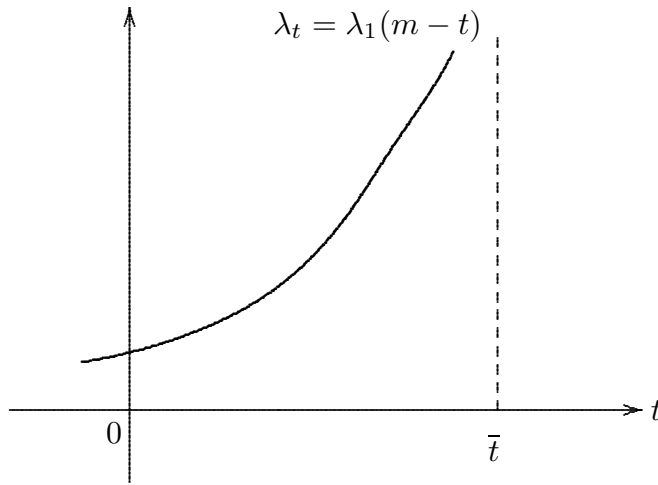


Figure 5.2

Step 3: The next lemma asserts the *uniqueness* of a positive eigenvalue of $L^{-1}M$ with a positive eigenfunction (see [12, Lemma 6]):

Lemma 5.3. *The operator $L^{-1}M : Y \rightarrow Y$ has a unique positive eigenvalue $\mu_1(m) = 1/\lambda_1(m)$ with a positive eigenfunction.*

Proof. Assume, to the contrary, that there is another positive eigenvalue μ_0 of $L^{-1}M$ with a positive eigenfunction v :

$$(L^{-1}M)v = \mu_0 v. \quad (5.8)$$

(i) First, we show that

$$0 < \mu_0 < \mu_1(m) = \frac{1}{\lambda_1(m)}. \quad (5.9)$$

Indeed, we have, by assertion (5.8),

$$\begin{aligned} (L^{-1}M)v = \mu_0 v, \quad v > 0 &\iff Lv = \frac{1}{\mu_0}Mv, \quad v > 0 \\ &\iff \left(L + \frac{1}{\mu_0}\right)v = \frac{1}{\mu_0}(M+1)v, \quad v > 0 \\ &\iff \mu_0 v = \left(L + \frac{1}{\mu_0}\right)^{-1}(M+1)v, \quad v > 0 \\ &\iff v = \frac{1}{\mu_0}K_{1/\mu_0}v, \quad v > 0. \end{aligned}$$

Hence, applying Lemma 4.5 with

$$\alpha_0 := \frac{1}{\mu_0}, \quad w_0 := v,$$

we can find a constant $\lambda > 0$ and a function $u \in P_X \setminus \{0\}$ such that

$$\begin{cases} 0 < \lambda \leq \frac{1}{\mu_0}, \\ Lu = \lambda Mu. \end{cases}$$

This implies that

$$\lambda \in \Lambda(m),$$

so that

$$\lambda_1(m) \leq \lambda \leq \frac{1}{\mu_0},$$

or equivalently,

$$\mu_0 \leq \frac{1}{\lambda_1(m)} = \mu_1(m).$$

Therefore, we have proved assertion (5.9), since $\mu_0 \neq \mu_1(m)$.

(ii) By assertion (5.9), we obtain that

$$\lambda_0 := \frac{1}{\mu_0} = \lambda_1(m - t_0)$$

for some $t_0 \in (0, \bar{t})$ (see Figure 5.3). Hence it follows that there exists a function $w \in \text{Int}(P_Y)$ such that

$$Lw = \lambda_1(m - t_0)(M - t_0)w, \quad (5.10)$$

or equivalently,

$$\mu_0 w = L^{-1}(M - t_0)w, \quad \mu_0 = \frac{1}{\lambda_1(m - t_0)}.$$

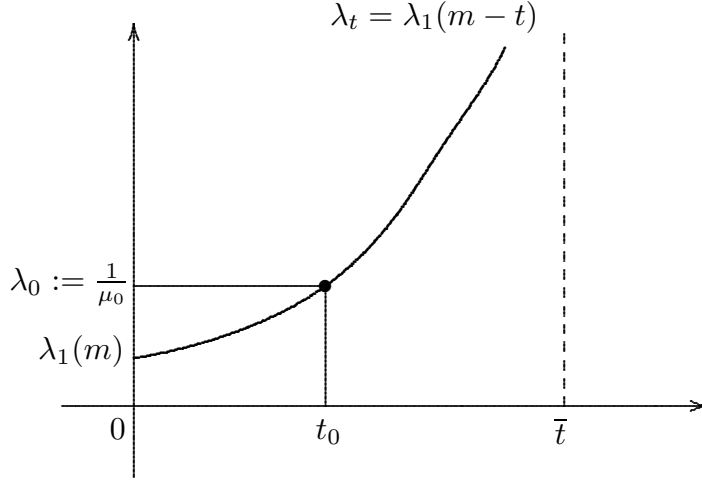


Figure 5.3

Then we have, by assertions (5.8) and (5.10),

$$\begin{aligned} \left(L + \frac{2}{\mu_0}\right)(\mu_0 v) &= (M + 2)v, \quad v > 0 \\ \left(L + \frac{2}{\mu_0}\right)(\mu_0 w) &= (M + 2 - t_0)w, \quad w > 0, \end{aligned}$$

and so

$$\begin{aligned} \left(L + \frac{2}{\mu_0}\right)^{-1} (M + 2)v &= \mu_0 v, \quad v > 0 \\ \left(L + \frac{2}{\mu_0}\right)^{-1} (M + 2 - t_0)w &= \mu_0 w, \quad w > 0. \end{aligned}$$

This implies that the eigenvalue problem

$$\left(L + \frac{2}{\mu_0}\right)^{-1} (M + 2 - t)u = \mu_0 u$$

has positive solutions v and w at $t = 0$ and $t = t_0$, respectively. Here it should be noticed that

$$m(x) + 2 > 0, \quad m(x) + 2 - t_0 > 0 \quad \text{on } \bar{\Omega}.$$

Therefore, applying the Kreĭn and Rutman theorem (Theorem 2.4) to our situation we obtain that

$$\begin{aligned} \operatorname{spr} \left(\left(L + \frac{2}{\mu_0} \right)^{-1} (M + 2) \right) &= \mu_1(m + 2) = \mu_0, \\ \operatorname{spr} \left(\left(L + \frac{2}{\mu_0} \right)^{-1} (M + 2 - t_0) \right) &= \mu_1(m + 2 - t_0) = \mu_0. \end{aligned}$$

In particular, we have

$$\lambda_1(m + 2) = \lambda_1(m + 2 - t_0) = \frac{1}{\mu_0}. \quad (5.11)$$

(iii) On the other hand, we recall that

$$\begin{aligned} \left(L + \frac{2}{\mu_0} \right) v &= \lambda(M + 2)v, \\ \left(L + \frac{2}{\mu_0} \right) w &= \lambda(M + 2 - t_0)w, \end{aligned}$$

and that

$$0 < m(x) + 2 - t_0 < m(x) + 2 \quad \text{on } \overline{\Omega}.$$

Hence, applying Proposition 5.1 to our situation we obtain that

$$0 < \lambda_1(m + 2) < \lambda_1(m + 2 - t_0).$$

This contradicts assertion (5.11).

The proof of Lemma 5.3 is complete. \square

Step 4: The next lemma asserts that $\mu_1(m) = 1/\lambda_1(m)$ is an eigenvalue of $L^{-1}M$ with *geometric multiplicity one* (see [12, Lemma 7]):

Lemma 5.4. *The eigenvalue $\mu_1(m) = 1/\lambda_1(m)$ is an eigenvalue of the operator $L^{-1}M : Y \rightarrow Y$ with geometric multiplicity one, and the geometric eigenspace is spanned by a positive function $\varphi_1 \in \operatorname{Int}(P_X)$.*

Proof. Assume that $v \in Y$ is an eigenfunction of $L^{-1}M$:

$$(L^{-1}M)v = \mu_1(m)v.$$

Then we have

$$\begin{aligned} (L^{-1}M)v = \mu_1(m)v &\iff Lv = \frac{1}{\mu_1(m)}Mv \\ &\iff \left(L + \frac{1}{\mu_1(m)} \right) v = \frac{1}{\mu_1(m)}(M + 1)v \\ &\iff \mu_1(m)v = \left(L + \frac{1}{\mu_1(m)} \right)^{-1} (M + 1)v \\ &\iff K_{\lambda_1(m)}v = \mu_1(m)v. \end{aligned} \quad (5.12)$$

However, by the Kreĭn and Rutman theorem (Theorem 2.4) and Lemma 5.3 it follows that

$$\text{spr}(K_{\lambda_1(m)}) = \mu_1(m) = \frac{1}{\lambda_1(m)}$$

is a unique eigenvalue of $K_{\lambda_1(m)}$ with a positive eigenfunction φ_1 . More precisely, we have

$$\begin{cases} K_{\lambda_1(m)}\varphi_1 = \mu_1(m)\varphi_1, \\ \varphi_1 \in \text{Int}(P_X), \end{cases}$$

and

$$N(\mu_1(m) - K_{\lambda_1(m)}) = \text{span}[\varphi_1]. \quad (5.13)$$

Therefore, by combining assertions (5.12) and (5.13) we obtain that $\mu_1(m)$ is an eigenvalue of the operator $L^{-1}M$ with geometric multiplicity one and the geometric eigenspace is spanned by $\varphi_1 \in \text{Int}(P_X)$.

The proof of Lemma 5.4 is complete. \square

Step 5: Finally, the next lemma asserts that $\mu_1(m) = 1/\lambda_1(m)$ is an eigenvalue of $L^{-1}M$ with *algebraic multiplicity one* (see [12, Lemma 8]):

Lemma 5.5. *The eigenvalue $\mu_1(m)$ of the operator $L^{-1}M : X \rightarrow X$ has algebraic multiplicity one.*

It should be emphasized that the principal eigenvalue $\mu_1(m)$ of the operator $L^{-1}M : Y \rightarrow Y$ coincides with the principal eigenvalue of the operator $L^{-1}M : X \rightarrow X$, since $L^{-1} : Y \rightarrow X$ is strongly positive.

Step 5-1: In order to prove Lemma 5.5, we need the following (see [12, Proposition 4]):

Proposition 5.6. *Assume that $m(x_0) > 0$ at some point $x_0 \in \Omega$. Then $\mu_1(m) := 1/\lambda_1(m)$ is the unique positive eigenvalue of $L^{*-1}M^* : Y^* \rightarrow Y^*$ having a positive eigenfunction. Here $L^{*-1}M^* = (ML^{-1})^*$ is the Banach space adjoint of ML^{-1} .*

Proof. (i) First, we show that $\mu_1 := \mu_1(m)$ is an eigenvalue of $L^{*-1}M^* : Y^* \rightarrow Y^*$ having a positive eigenfunction.

We remark that

$$\begin{aligned} Lu &= \lambda_1(m)Mu, \quad u > 0 \\ \iff \\ u &= \lambda_1(m)(L + \lambda_1(m))^{-1}(M + 1)u = \lambda_1(m)K_{\lambda_1(m)}u, \quad u > 0 \\ \iff \\ K_{\lambda_1(m)}u &= \mu_1(m)u, \quad u > 0. \end{aligned}$$

Therefore, applying the Kreĭn and Rutman theorem (Theorem 2.4) we obtain that $\mu_1(m) = 1/\lambda_1(m)$ is the principal eigenvalue of $K_{\lambda_1(m)}$ and of $(K_{\lambda_1(m)})^*$. Hence we can find an element $v^* \in P_{Y^*} \setminus \{0\}$ such that

$$(K_{\lambda_1(m)})^* v^* = \mu_1(m)v^*,$$

or equivalently,

$$(M + 1)^*(L + \lambda_1(m))^{*-1}v^* = \mu_1(m)v^*, \quad (5.14)$$

where

$$P_Y^* = \{u^* \in Y^* : \langle u^*, u \rangle \geq 0 \text{ for all } u \in P_Y\}.$$

If we let

$$u^* = (L + \lambda_1(m))^{*-1} v^*,$$

then we have, by formula (5.14),

$$\begin{cases} u^* \in P_Y^* \setminus \{0\}, \\ (M^* + 1)u^* = (M + 1)^* u^* = \mu_1(m) (L + \lambda_1(m))^* u^* = (\mu_1(m)L^* + 1)u^*, \end{cases}$$

and so

$$\begin{cases} u^* \in P_Y^* \setminus \{0\}, \\ \mu_1(m)u^* = L^{*-1} M^* u^*. \end{cases}$$

Indeed, it suffices to note that

$$\langle u^*, u \rangle = \langle v^*, (L + \lambda_1(m))^{-1} u \rangle > 0, \quad u > 0,$$

since $v^* \in P_Y^* \setminus \{0\}$ and $(L + \lambda_1(m))^{-1} u \gg 0$.

Moreover, it should be noticed that $\mu_1(m)$, as an eigenvalue of $L^{*-1} M^*$, has geometric multiplicity one. Indeed, we have

$$\begin{aligned} (L^{*-1} M^*)v^* &= \mu_1(m)v^*, \quad v^* \neq 0 \\ \iff \\ \mu_1(m)L^*v^* &= M^*v^*, \quad v^* \neq 0 \\ \iff \\ \mu_1(m)(L + \lambda_1(m))^*v^* &= (M + 1)^*v^*, \quad v^* \neq 0, \end{aligned}$$

and also

$$\mu_1(m)w^* = ((M + 1)^*(L^* + \lambda_1(m))^{-1})w^* = (K_{\lambda_1(m)})^*w^*,$$

with

$$w^* = (L + \lambda_1(m))^*v^* \neq 0.$$

This proves that

$$\dim N \left(L^{*-1} M^* - \mu_1(m) \right) = \dim N \left((K_{\lambda_1(m)})^* - \mu_1(m) \right) = 1.$$

(ii) Before continuing the proof of Proposition 5.6, we need the following (see [22, Appendix]):

Claim 5.7. *Let $A : Y \rightarrow Y$ be an irreducible, compact positive operator and let ν be an eigenvalue of $A^* : Y^* \rightarrow Y^*$ with a positive eigenfunction $u^* : A^*u^* = \nu u^*$. Then it follows that ν coincides with the principal eigenvalue ν_1 of A .*

Proof. If $w \gg 0$ is an eigenfunction of A associated with ν_1 :

$$Aw = \nu_1 w,$$

then we have

$$\nu \langle u^*, w \rangle = \langle A^* u^*, w \rangle = \langle u^*, Aw \rangle = \nu_1 \langle u^*, w \rangle.$$

This implies that $\nu = \nu_1$, since $\langle u^*, w \rangle > 0$. \square

(iii) Finally, we prove the uniqueness of a positive eigenvalue of $L^{*-1} M^*$ with a positive eigenfunction.

Assume that $\mu > 0$ is an eigenvalue of $L^{*-1} M^*$ with a positive eigenfunction $u^* \in P_Y^* \setminus \{0\}$:

$$(L^{*-1} M^*) u^* = \mu u^*.$$

Since we have

$$\mu u^* = (L + \lambda)^{*-1} (M + 1)^* u^*, \quad \lambda = \frac{1}{\mu},$$

it follows from an application of Claim 5.7 with

$$A := (M + 1)(L + \lambda)^{-1}$$

that μ is the principal eigenvalue of $(M + 1)(L + \lambda)^{-1}$, so that there exists a function $w > 0$ such that

$$(M + 1)(L + \lambda)^{-1} w = \mu w.$$

Hence, by letting

$$u := (L + \lambda)^{-1} w,$$

we obtain that

$$\begin{cases} (L^{-1} M) u = \mu u, \\ u > 0. \end{cases}$$

By Lemma 5.3, this proves that $\mu = \mu_1(m)$. \square

Step 5-2: In the proof of Lemma 5.5, we make use of *analytic perturbation theory* (see [15]). Let T be a compact linear operator in a complex Banach space Z . Assume that μ is a non-zero eigenvalue of T with geometric multiplicity one and algebraic multiplicity $r \geq 1$. Let u and w^* be the unique (up to scalar factors) eigenvectors of T and T^* for μ , respectively. Here $T^* : Z^* \rightarrow Z^*$ is the Banach space adjoint of T . It should be emphasized that T and T^* have the same non-zero eigenvalues.

The next lemma plays a fundamental role in the proof of Lemma 5.5 (see [12, Lemma 8a]):

Lemma 5.8. *Let T be a compact linear operator in a complex Banach space Z , and let $\mu \neq 0$ be an eigenvalue of T with geometric multiplicity one and algebraic multiplicity $r \geq 1$. If V is a bounded linear operator in Z , then we let*

$$T(\kappa) := T + \kappa V, \quad \kappa \in \mathbf{C}.$$

If u and w^ are eigenvectors of T and T^* for μ , respectively, and if we have*

$$\langle w^*, Vu \rangle \neq 0, \tag{5.15}$$

then the operator $T(\kappa)$ has exactly r distinct eigenvalues $\mu(\kappa)$ near μ for $|\kappa|$ sufficiently small, given by the formula

$$\mu(\kappa) = \mu + (ar)^{1/r} + o(|\kappa|)^{1/r}, \tag{5.16}$$

where $a \neq 0$ is a constant, depending on $\langle w^*, Vu \rangle$, made precise in the proof.

Proof. First, since the eigenvalue μ has geometric multiplicity one, we can find a canonical basis $\{u_1, u_2, \dots, u_r\}$ of the algebraic eigenspace of T for μ such that

$$\begin{aligned} u_k &= (T - \mu)^{r-k} u_r, \quad k = 1, 2, \dots, r, \\ (T - \mu)u_1 &= 0. \end{aligned}$$

Namely, the operator T can be expressed, with respect to the basis $\{u_1, u_2, \dots, u_r\}$, in the form

$$T \sim \begin{pmatrix} \mu & 1 & & 0 \\ & \mu & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu \end{pmatrix}.$$

Without loss of generality, we may assume that $u_1 = u$.

Similarly, let $\{u_1^*, u_2^*, \dots, u_r^*\}$ be the dual basis of the algebraic eigenspace of T^* for μ :

$$\langle u_j^*, u_k \rangle = \delta_{jk}.$$

Then it is easy to verify that

$$\begin{aligned} u_j^* &= (T^* - \mu)^{j-1} u_1^*, \quad j = 1, 2, \dots, r, \\ (T^* - \mu)u_r^* &= 0. \end{aligned}$$

The operator T^* can be expressed, with respect to the basis $\{u_r^*, u_{r-1}^*, \dots, u_1^*\}$, in the form

$$T^* \sim \begin{pmatrix} \mu & 1 & & 0 \\ & \mu & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu \end{pmatrix}.$$

Hence it follows that $u_r^* = \sigma w^*$ for some $\sigma \neq 0$.

Analytic perturbation theory (see [15, Chapter VII, Section 1.3]) asserts that the eigenvalues $\mu(\kappa)$ of $T(\kappa)$ near μ are the roots of the equation

$$\det [\langle u_j^*, (T(\kappa) - \mu(\kappa))u_k \rangle + O(|\kappa|^2)] = 0. \quad (5.17)$$

Since we have

$$\langle u_j^*, (T - \mu)u_k \rangle = \delta_{jk-1},$$

we can rewrite equation (5.17) as follows:

$$\begin{vmatrix} \mu - \mu(\kappa) + \kappa V_{11} & 1 + \kappa V_{12} & \dots & \kappa V_{1r} \\ \kappa V_{21} & \mu - \mu(\kappa) + \kappa V_{22} & \dots & \kappa V_{2r} \\ \vdots & & \ddots & \vdots \\ \kappa V_{r1} & \kappa V_{r2} & \dots & \mu - \mu(\kappa) + \kappa V_{rr} \end{vmatrix} + O(|\kappa|^2) = 0, \quad (5.18)$$

where

$$V_{jk} := \langle u_j^*, Vu_k \rangle.$$

However, it should be noticed that

$$\mu - \mu(\kappa) = o(1) \quad \text{as } \kappa \rightarrow 0.$$

Therefore, it follows from formula (5.18) that

$$\mu(\kappa) = \mu + (V_{r1}\kappa)^{1/r} + o\left(|\kappa|^{1/r}\right) \quad \text{as } \kappa \rightarrow 0.$$

This proves assertion (5.16), since we have, condition (5.15),

$$V_{r1} = \langle u_r^*, V u_1 \rangle = \sigma \langle w^*, V u \rangle \neq 0.$$

The proof of Lemma 5.8 is complete. \square

Step 5-3: *Proof of Lemma 5.5.* We let

$$T := L^{-1} M, \quad \mu := \mu_1(m),$$

and assume, to the contrary, that the eigenvalue μ of T has algebraic multiplicity $r \geq 2$.

First, we choose a canonical basis $\{u_1, u_2, \dots, u_r\}$ of the algebraic eigenspace of T for μ , just as in the proof of Lemma 5.8, and let $\{u_1^*, u_2^*, \dots, u_r^*\}$ be the dual basis of the algebraic eigenspace of T^* for μ . Here it should be noticed that the bases $\{u_k\}$ and $\{u_j^*\}$ may be chosen to be *real*.

Now let u be an eigenfunction of T for μ . By virtue of Lemma 5.4, we may assume that

$$u \gg 0,$$

and that

$$u_1 = u.$$

If we let

$$u^* := L^{*-1} u_r^*,$$

then we obtain that

$$\left(L^{*-1} M^*\right) u^* = \mu u^*.$$

Indeed, we have

$$\begin{aligned} L^* u^* &= u_r^* = \frac{1}{\mu} T^* u_r^* \\ &= \frac{1}{\mu} (L^{-1} M)^* u_r^* \\ &= \frac{1}{\mu} M^* (L^{-1})^* u_r^* \\ &= \frac{1}{\mu} M^* L^{*-1} u_r^* \\ &= \frac{1}{\mu} M^* u^*. \end{aligned}$$

Hence, applying Proposition 5.6 we obtain that

$$u^* > 0 \quad \text{or} \quad -u^* > 0.$$

This implies that

$$\begin{aligned} V_{r1} &:= -\langle u_r^*, L^{-1}u \rangle \\ &= -\langle L^{*-1}u_r^*, u \rangle \\ &= -\langle u^*, u \rangle \begin{cases} < 0 & \text{if } u^* > 0, \\ > 0 & \text{if } -u^* > 0, \end{cases} \end{aligned}$$

since $u \gg 0$.

Therefore, by applying Lemma 5.8 with

$$\begin{aligned} Z &:= \hat{Y}, \\ T &:= \hat{L}^{-1} \hat{M}, \\ V &:= -\hat{L}^{-1}, \\ T(\kappa) &:= \hat{S}(\kappa) = \hat{L}^{-1}(\hat{M} - \kappa), \quad \kappa \in \mathbf{C}, \\ u &:= u_1, \\ w^* &:= u_r^*, \end{aligned}$$

we conclude that the operator $\hat{S}(t)$ has exactly r eigenvalues roughly given by the formula

$$\mu_1(m) + (V_{r1} t)^{1/r} \tag{5.19}$$

for sufficiently small real $t \neq 0$.

(a) If r is even, then it follows from formula (5.19) that none of these eigenvalues are real for sufficiently small real $t \neq 0$ if the sign of t is properly chosen. This contradicts Lemma 5.2 which asserts that the operator $S(t)$ has a real eigenvalue $\mu_1(m - t)$ close to $\mu_1(m)$ (see Figure 5.2).

(b) If r is odd and $r \geq 3$, then it follows from formula (5.19) that one of these eigenvalues is real and all the others are non-real complex numbers nearly symmetrically distributed on a small circle about $\mu_1(m)$, for sufficiently small real $t \neq 0$. Hence, if the sign of t is properly chosen, the real eigenvalue becomes smaller than $\mu_1(m)$ and some of the complex eigenvalues have real parts larger than $\mu_1(m)$. This contradicts Part (i) of Theorem 1.1 which asserts that the real one must have the largest real part.

The proof of Lemma 5.5 is complete. \square

Now the proof of Part (ii) of Theorem 1.2 is complete. \square

6. Proof of Theorem 1.3

The proof of Theorem 1.3 may be carried out by using Theorem 1.2, the Crandall and Rabinowitz local bifurcation theorem ([7]) and the Rabinowitz global bifurcation theorem ([20]), just as in Hess and Kato [12, Theorem 2] and Hess [11, Theorem 27.1]. Our proof is divided into six steps.

Step 1: Let $X := D(L)$. First, we prove that if a pair $(\lambda, u) \in \mathbf{R} \times X$ is a positive solution of problem (1.1), then it follows that $\lambda > \lambda_1(m)$.

By rescaling, we may assume that

$$|m(x)| < 1 \quad \text{on } \overline{\Omega}.$$

By using the resolvent L^{-1} for problem (3.1), we shall transform problem (1.1) into a nonlinear operator equation in an appropriate ordered Banach space (cf. [1]). It follows from an application of Proposition 4.3 that a function u is a solution of problem (1.1) if and only if it satisfies the equation

$$u = \lambda L^{-1} (m(x)u - h(x)u^2) \quad \text{in } X. \quad (6.1)$$

By assertion (i) of Theorem 1.2, we can find a function $\varphi_1 \in \text{Int}(P_X)$ such that

$$L\varphi_1 = \lambda_1(m)M\varphi_1 \quad \text{in } \Omega.$$

Now assume that a pair (λ, u) with $\lambda > 0$ and $u \in P_X \setminus \{0\}$ satisfies the operator equation (6.1). Then we have, by the strong positivity of the Green operator,

$$u \in \text{Int}(P_X).$$

Indeed, if we let

$$d = \max_{\substack{x \in \overline{\Omega} \\ 0 \leq s \leq \|u\|_{C(\overline{\Omega})}}} |m(x) - h(x)s| + 1,$$

then it follows that

$$\begin{aligned} (\lambda d + L)u &= \lambda(du + F(u)) \\ &= \lambda u(m(x) - h(x)u + d) > 0 \quad \text{a.e. in } \Omega. \end{aligned}$$

Hence, by the strong positivity of the Green operator $(\lambda d + L)^{-1}$ we obtain that

$$u = \lambda(\lambda d + L)^{-1}(F(u) + du) \in \text{Int}(P_X).$$

Moreover, we have

$$\begin{aligned} (\lambda + L)u &= \lambda(m(x) + 1)u - \lambda h(x)u^2 \\ &< \lambda(m(x) + 1)u \quad \text{a.e. in } \Omega. \end{aligned}$$

This implies that

$$\lambda K_\lambda u - u = (\lambda + L)^{-1} (\lambda(M + 1)u - (\lambda + L)u) \in \text{Int}(P_X).$$

Therefore, applying Lemma 4.5 with

$$\alpha_0 := \lambda, \quad w_0 := u, \quad \lambda := \lambda_1(m),$$

we obtain that

$$\lambda > \lambda_1(m).$$

Step 2: Secondly, we prove a uniqueness result for positive solutions of problem (1.1), which implies that the unbounded continuum \mathcal{C}^+ is actually an *arc*:

Lemma 6.1. *Problem (1.1) has at most one positive solution $u(\lambda)$ for any $\lambda > \lambda_1(m)$.*

Proof. First, we introduce a mapping

$$H(\lambda, v) : \mathbf{R}^+ \times X \longrightarrow X$$

defined by the formula

$$H(\lambda, v) = \lambda(\lambda + L)^{-1}(F(v) + v), \quad \lambda > 0, \quad v \in X.$$

Then it is easy to see that

$$Lu = \lambda F(u) \quad \text{in } Y$$

if and only if

$$u = H(\lambda, u) \quad \text{in } X.$$

Now let $\lambda > \lambda_1(m)$ and assume, to the contrary, that $u_1(x)$ and $u_2(x)$ are two different positive solutions of problem (1.1), that is,

$$\begin{aligned} u_1 &\in \text{Int}(P_X), \quad Lu_1 = \lambda F(u_1), \\ u_2 &\in \text{Int}(P_X), \quad Lu_2 = \lambda F(u_2), \\ u_1 &\neq u_2. \end{aligned}$$

Then we have

$$\begin{aligned} u_1 &= H(\lambda, u_1), \\ u_2 &= H(\lambda, u_2). \end{aligned}$$

By rescaling, we may assume that

$$m(x) - 2h(x)s + 1 > 0, \quad x \in \overline{\Omega}, \quad 0 \leq s \leq \overline{s},$$

where

$$\overline{s} = \max \left\{ \|u_1\|_{C(\overline{\Omega})}, \|u_2\|_{C(\overline{\Omega})} \right\} + 1.$$

This implies that the function $s \mapsto H(\lambda, s)$ is increasing on the interval $[0, \overline{s}]$.

Since $u_1, u_2 \in \text{Int}(P_X)$ and $u_1 \neq u_2$, we can find a constant $0 < \tau < 1$ such that

$$u_1 - \tau u_2 \in \partial P_X.$$

However, by the strong positivity of $(\lambda + L)^{-1}$ it follows that

$$\begin{aligned} u_1 - \tau u_2 &= H(\lambda, u_1) - \tau H(\lambda, u_2) \\ &\geq H(\lambda, \tau u_2) - \tau H(\lambda, u_2) \\ &= \tau \lambda (\lambda + L)^{-1} ((m(x) + 1)u_2 - h(x)\tau u_2^2) \\ &\quad - \tau \lambda (\lambda + L)^{-1} ((m(x) + 1)u_2 - h(x)u_2^2) \\ &= \tau \lambda (\lambda + L)^{-1} (h(x)(1 - \tau)u_2^2) \in \text{Int}(P_X), \end{aligned}$$

since $h(x)(1 - \tau)u_2^2 > 0$ in Ω . This contradicts the choice of the constant τ : $u_1 - \tau u_2 \in \partial P_X$.

The proof of Lemma 6.1 is complete. \square

Step 3: Thirdly, the next lemma proves the existence of positive solutions of problem (1.1) bifurcating at the point $(\lambda_1(m), 0)$:

Lemma 6.2. *There exists an unbounded continuum \mathcal{C}^+ of positive solutions of equation (6.1) emanating from $(\lambda_1(m), 0)$.*

Proof. (1) Just as in the proof of Hess and Kato [12, Theorem 2], we extend the function

$$f(x, s) = m(x)s - h(x)s^2$$

as an *odd* function in the variable s as follows:

$$\tilde{f}(x, s) = \begin{cases} m(x)s - h(x)s^2 & \text{if } s > 0, \\ m(x)s + h(x)s^2 & \text{if } s \leq 0. \end{cases}$$

Then we associate with the function $\tilde{f}(x, s)$ the Nemytskii operator $\tilde{F}(u)$ defined by the formula

$$\tilde{F}(u) = \tilde{f}(x, u(x)), \quad x \in \overline{\Omega},$$

and consider instead of equation (6.1) the following equation:

$$u = \lambda L^{-1} \left(\tilde{F}(u) \right) \quad \text{in } X. \quad (6.2)$$

It should be noticed that u is a solution of equation (6.2) if and only if $-u$ is a solution; hence we may identify positive solutions with negative solutions in what follows.

(2) Now we show that the Crandall and Rabinowitz local bifurcation theorem (Theorem 2.5) may be employed to assert the existence of the continuum of non-trivial solutions of problem (1.1) emanating from $(\lambda_1(m), 0)$, which can be expressed as the union \mathcal{C} of two subcontinua intersecting at $(\lambda_1(m), 0)$ (cf. Deimling [8, Corollary 29.1]).

We apply Theorem 2.5 with

$$\begin{aligned} X &:= D(L), \quad Y := C(\overline{\Omega}), \\ F(t, x) &:= Lu - \lambda F(u) = Lu - \lambda (m(x)u - h(x)u^2), \\ F_x(0, 0) &:= L - \lambda_1(m) M, \\ F_{tx}(0, 0) &:= -M, \\ x_0 &:= \varphi_1(x). \end{aligned}$$

To do this, it suffices to verify the following two assertions:

$$(2a) \dim N(L - \lambda_1(m) M) = \text{codim } R(L - \lambda_1(m) M) = 1.$$

$$(2b) M\varphi_1 \notin R(L - \lambda_1(m) M).$$

Proof of Assertion (2a): First, since $L : X \rightarrow Y$ is an isomorphism and since $\lambda_1(m) M : X \rightarrow Y$ is compact, we find that $L - \lambda_1(m) M$ is a Fredholm operator with index zero

$$\text{ind}(L - \lambda_1(m) M) = \text{ind}(L) = 0,$$

that is,

$$\dim N(L - \lambda_1(m) M) = \text{codim } R(L - \lambda_1(m) M). \quad (6.3)$$

However, we have, by the Kreĭn and Rutman theorem (Theorem 2.4),

$$\dim N(L - \lambda_1(m) M) = 1. \quad (6.4)$$

Indeed, it suffices to note that

$$(L - \lambda_1(m)M)u = 0 \iff K_{\lambda_1(m)}u = \frac{1}{\lambda_1(m)}u,$$

and that $1/\lambda_1(m) = \text{spr}(K_{\lambda_1(m)})$ is a *simple* eigenvalue of $K_{\lambda_1(m)}$, since $\lambda_1(m)$ is an eigenvalue of equation (1.3) having a positive eigenfunction $\varphi_1 \in \text{Int}(P_Y)$.

Therefore, combining assertions (6.3) and (6.4) we obtain that

$$\dim N(L - \lambda_1(m)M) = \text{codim } R(L - \lambda_1(m)M) = 1.$$

Proof of Assertion (2b): Secondly, since we have

$$N(L - \lambda_1(m)M) = \text{span}[\varphi_1],$$

we obtain that

$$\begin{aligned} (L - \lambda_1(m)M)\varphi_1 = 0 &\iff M\varphi_1 = \frac{1}{\lambda_1(m)}L\varphi_1 = \mu_1(m)L\varphi_1 \\ &\iff (L^{-1}M)\varphi_1 = \mu_1(m)\varphi_1. \end{aligned}$$

However, by Lemma 5.5 it follows that $\mu_1(m)$ is an eigenvalue of $L^{-1}M$ with algebraic multiplicity one. This implies that

$$\varphi_1 \notin R(L^{-1}M - \mu_1(m)),$$

so that

$$M\varphi_1 = \mu_1(m)L\varphi_1 \notin R(L - \lambda_1(m)M).$$

(3) Moreover, by Part (ii) of Proposition 4.3 and Lemma 5.5 it follows that these subcontinua \mathcal{C} are *locally* the strictly positive and the strictly negative solutions of equation (6.2).

Indeed, assume, to the contrary, that there exists a sequence (λ_j, u_j) , with $\lambda_j > 0$ and $u_j \in X$, such that

$$\begin{aligned} u_j &= \lambda_j L^{-1} \left(\tilde{F}(u_j) \right), \\ \lambda_j &\rightarrow \lambda_1(m), \\ u_j &\rightarrow 0 \quad \text{in } X, \\ u_j &\notin \text{Int}(P_X). \end{aligned}$$

If we let

$$v_j = \frac{u_j}{\|u_j\|_X},$$

then it follows that

$$\begin{aligned} v_j &\notin \text{Int}(P_X), \\ \|v_j\|_X &= 1, \\ v_j &= \frac{\lambda_j}{\|u_j\|_X} L^{-1} \left(\tilde{F}(u_j) \right). \end{aligned}$$

By the compactness of L^{-1} , we can choose a subsequence $\{v_{j'}\}$ which converges to some function v in X . Therefore, passing to the limit we obtain that

$$\begin{aligned} v &\notin \text{Int}(P_X), \\ \|v\|_X &= 1, \end{aligned} \tag{6.5}$$

and that

$$v = \lambda_1(m)L^{-1}(m(x)v),$$

or equivalently,

$$\mathcal{L}v = \lambda_1(m)m(x)v.$$

Since $\lambda_1(m)$ is a *simple* eigenvalue of problem (1.2) having an eigenfunction in $\text{Int}(P_X)$, it follows that

$$v \in \text{Int}(P_X).$$

This contradicts condition (6.5).

(4) We show that these subcontinua \mathcal{C} are *globally* the strictly positive and the strictly negative solutions of equation (6.2) (see Figure 6.1).

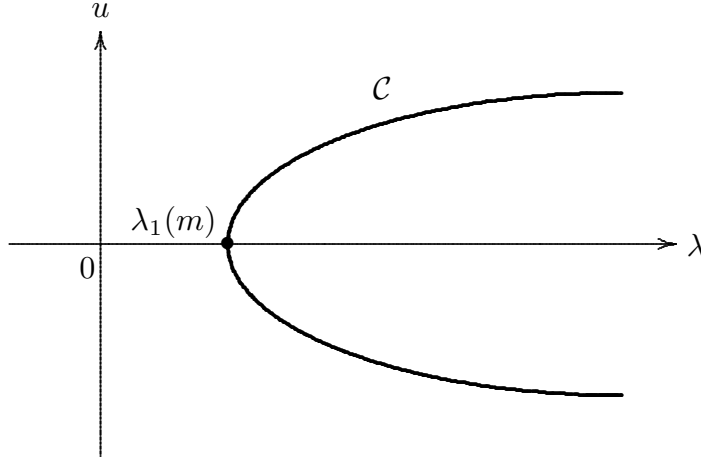


Figure 6.1

Indeed, assume, to the contrary, that there exists a point (λ_0, u_0) of these subcontinua such that

$$\begin{aligned} \lambda_0 &> 0, \\ u_0 &\in \partial P_X, \\ u_0 &> 0, \\ u_0 &= \lambda_0 L^{-1}(F(u_0)). \end{aligned} \tag{6.6}$$

If we let

$$d_0 = \max_{\substack{x \in \overline{\Omega} \\ 0 \leq s \leq \|u_0\|_{C(\overline{\Omega})}}} |m(x) - h(x)s| + 1,$$

then it follows that

$$\begin{aligned} (\lambda_0 d_0 + L) u_0 &= \lambda_0 (d_0 u_0 + F(u_0)) \\ &= \lambda_0 u_0 (m(x) - h(x)u_0 + d_0) > 0 \quad \text{a.e. in } \Omega. \end{aligned}$$

Hence we have, by the strong positivity of $(\lambda_0 d_0 + L)^{-1}$,

$$u_0 = \lambda_0 (\lambda_0 d_0 + L)^{-1} (F(u_0) + d_0 u_0) \in \text{Int}(P_X).$$

However, this contradicts condition (6.6).

On the other hand, it is clear that equation (6.2) has no non-trivial solutions for $\lambda = 0$.

(5) Finally, the Rabinowitz global bifurcation theorem [20, Theorem 1.10] asserts that the subcontinuum \mathcal{C}^+ of positive solutions emanating from $(\lambda_1(m), 0)$ is either unbounded or contains another bifurcation point $(\lambda, 0)$ with $\lambda \neq \lambda_1(m)$.

However, just as in Step (4) we can prove that the subcontinuum \mathcal{C}^+ can not contain a point $(\lambda, 0)$ with $\lambda \neq \lambda_1(m)$; hence \mathcal{C}^+ must be *unbounded* (cf. [8, Theorem 29.2]).

The proof of Lemma 6.2 is now complete. \square

Step 4: By using the implicit function theorem, we show that there exists a *critical value* $\lambda^* \in (\lambda_1(m), +\infty]$ such that we can parametrize the bifurcation solution curve $(\lambda, u(\lambda))$ by λ , $\lambda_1(m) < \lambda < \lambda^*$, as a C^1 curve (cf. Hess [11, Theorem 27.1]):

Lemma 6.3. *There exists a constant $\lambda^* \in (\lambda_1(m), +\infty]$ such that we have a positive solution $(\lambda, u(\lambda))$ of the equation $u = H(\lambda, u)$ for all $\lambda \in (\lambda_1(m), \lambda^*)$.*

Proof. By rescaling, we may assume again that

$$m(x) - 2h(x)s + 1 > 0, \quad x \in \overline{\Omega}, \quad 0 \leq s \leq \|u\|_{C(\overline{\Omega})} + 1.$$

Then, applying Proposition 4.3 to our situation we obtain that the Fréchet derivative

$$H_v(\lambda, u) = \lambda(\lambda + L)^{-1}(F'(u) + I) : X \longrightarrow X$$

at (λ, u) is strongly positive and compact.

The next claim guarantees the *bijectivity* of the Fréchet derivative $H_v(\lambda, u)$:

Claim 6.4. *If $r^* = \text{spr}(H_v(\lambda, u))$ is the principal eigenvalue of $H_v(\lambda, u)$, then it follows that $0 < r^* < 1$.*

Proof. Assume, to the contrary, that

$$r^* \geq 1.$$

By the Kreĭn and Rutman theorem (Theorem 2.4), it follows that there exists a function $w \in \text{Int}(P_X)$ such that

$$H_v(\lambda, u)w = r^* w.$$

However, we can find a constant $t_0 > 0$ such that

$$u - t_0 r^* w \in \partial P_X, \tag{6.7}$$

since $u, w \in \text{Int}(P_X)$. Then we have

$$H(\lambda, u - t_0 r^* w) \in P_X. \quad (6.8)$$

Indeed, it suffices to note that the function $H(\lambda, \cdot)$ is increasing and $H(\lambda, 0) = 0$.

On the other hand, it follows that

$$\begin{aligned} & u - t_0 r^* w \\ &= H(\lambda, u) - t_0 H_v(\lambda, u) w \\ &= \lambda(\lambda + L)^{-1} ((m(x)u - h(x)u^2 + u) - t_0(m(x) - 2h(x)u + 1)w) \\ &= H(\lambda, u - t_0 w) + \lambda t_0^2 (\lambda + L)^{-1} (h(x)w^2) \\ &\geq H(\lambda, u - t_0 r^* w) + \lambda t_0^2 (\lambda + L)^{-1} (h(x)w^2), \end{aligned} \quad (6.9)$$

since $u - t_0 w \geq u - t_0 r^* w$ for $r^* \geq 1$. Moreover, it follows that

$$\lambda t_0^2 (\lambda + L)^{-1} (h(x)w^2) \in \text{Int}(P_X), \quad (6.10)$$

since $h(x)w^2 > 0$ in Ω .

Therefore, combining assertions (6.8), (6.9) and (6.10) we obtain that

$$u - t_0 r^* w \in \text{Int}(P_X).$$

This contradicts condition (6.7). \square

By Claim 6.4, it follows that the Fréchet derivative $I - H_v(\lambda, u)$ is invertible in X . Hence, by using the implicit function theorem we can find a positive bifurcation solution curve $(\lambda, \tilde{u}(\lambda))$ of the equation $u = H(\lambda, u)$ for all $\lambda \in (\lambda_1(m), \lambda^*)$.

The proof of Lemma 6.3 is complete. \square

Step 5: Now we prove the *uniform estimate*

$$\max_{\overline{\Omega}} |\tilde{u}(\lambda)| \leq \frac{\max_{\overline{\Omega}} m}{\min_{\overline{\Omega}} h}. \quad (1.5)$$

If we let

$$w(x) \equiv \ell := \frac{\max_{\overline{\Omega}} m}{\min_{\overline{\Omega}} h},$$

then we have

$$\begin{aligned} \mathcal{L}w - \lambda m(x)w + \lambda h(x)w^2 &= c(x)\ell - \lambda m(x)\ell + \lambda h(x)\ell^2 \\ &\geq \lambda \ell (h(x)\ell - m(x)) \\ &\geq 0 \quad \text{in } \Omega, \end{aligned}$$

and also

$$w > 0 \quad \text{on } \partial\Omega.$$

This implies that the function $w(x)$ is a *supersolution* of problem (1.1).

Therefore, applying a comparison theorem based on the maximum principle (Theorem 3.4) we obtain the uniform estimate (1.5)

$$0 \leq \tilde{u}(\lambda) \leq \ell \quad \text{on } \overline{\Omega}.$$

Step 6: Finally, it remains to prove that $\lambda^* = +\infty$.

Assume, to the contrary, that

$$\lambda^* < +\infty.$$

Then, by the uniform estimate (1.5) it follows that

$$\begin{aligned} \|L\tilde{u}(\lambda)\|_{C(\overline{\Omega})} &= \lambda \|m(x)\tilde{u}(\lambda) - h(x)\tilde{u}(\lambda)^2\|_{C(\overline{\Omega})} \\ &\leq \lambda \left(\ell \|m\|_{C(\overline{\Omega})} + \ell^2 \|h\|_{C(\overline{\Omega})} \right) \\ &\leq \lambda^* \ell \left(\|m\|_{C(\overline{\Omega})} + \ell \|h\|_{C(\overline{\Omega})} \right). \end{aligned}$$

This implies that

$$\begin{aligned} \|\tilde{u}(\lambda)\|_X &= \|\tilde{u}(\lambda)\|_{C_0(\overline{\Omega})} + \|L\tilde{u}(\lambda)\|_{C(\overline{\Omega})} \\ &\leq \ell \left(1 + \lambda^* \|m\|_{C(\overline{\Omega})} \right) + \lambda^* \ell^2 \|h\|_{C(\overline{\Omega})}, \quad 0 < \lambda < \lambda^*. \end{aligned}$$

Thus, by the compactness argument we may assume that there exist a subsequence $(\lambda_{j'}, u_{j'})$, with $\lambda_{j'} > 0$ and $u_{j'} \in X$, and an element $u^* \in X$ such that

$$\begin{aligned} u_{j'} &= \lambda_{j'} L^{-1} (F(u_{j'})), \\ \lambda_{j'} &\rightarrow \lambda^*, \\ u_{j'} &\rightarrow u^* \quad \text{in } X. \end{aligned}$$

However, it should be noticed that the point $(\lambda_1(m), 0)$ is the only bifurcation point for positive solutions from the line of trivial solutions.

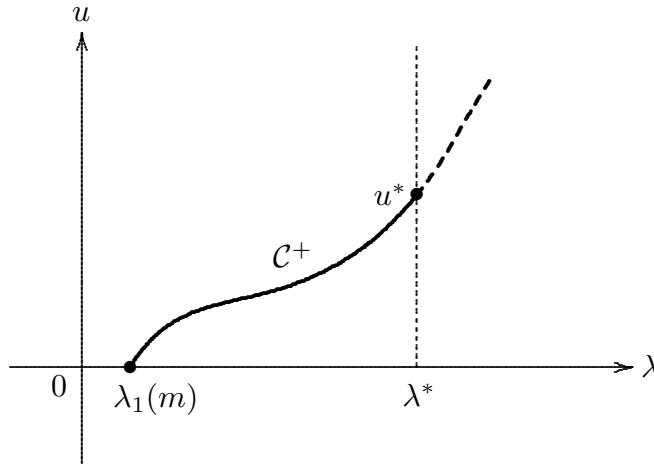


Figure 6.2

Therefore, we obtain that the limit point (λ^*, u^*) is a positive solution of problem (1.1). The implicit function theorem asserts that the positive solution curve \mathcal{C}^+ can be continued beyond the point (λ^*, u^*) (see Figure 6.2), just as in the proof of Lemma 6.3. This contradicts the definition of the critical value λ^* .

Now the proof of Theorem 1.3 is complete. \square

7. Concluding remarks

The logistic *Neumann problem* may be treated just as in Senn [23] if we make use of the results of Senn and Hess [24], Maugeri and Palagachev [18] and Lieberman [17]. In other words, we can generalize Senn and Hess [24, Theorems 2 and 3] and Senn [23, Theorem 2.4] to the *VMO* case, by using a generation theorem for Feller semigroups with Ventcel' boundary conditions which will be proved in the forthcoming paper.

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